

STEENROD HOMOTOPY

SERGEY A. MELIKHOV

ABSTRACT. Steenrod homotopy theory is a natural framework for doing algebraic topology on general spaces in terms of algebraic topology of polyhedra; or from a different viewpoint, it studies the topology of the \lim^1 functor (for inverse sequences of groups). This paper is primarily concerned with the case of compacta, in which Steenrod homotopy coincides with strong shape. An attempt is made to simplify the foundations of the theory and to clarify and improve some of its major results.

We use geometric tools such as Milnor’s telescope compactification, comanifolds (=mock bundles) and the Pontryagin–Thom Construction, to obtain new simple proofs for results by Barratt–Milnor; Cathey; Dydak; Dydak–Segal; Eda–Kawamura; Edwards–Geoghegan; Fox; Geoghegan–Krasinkiewicz; Jussila; Krasinkiewicz–Minc; Mardešić; Mittag-Leffler–Bourbaki; and for three unpublished results by Shchepin. An error in Lisitsa’s proof of the “Hurewicz theorem in Steenrod homotopy” is corrected. It is shown that over compacta, R. H. Fox’s overlayings are equivalent to I. M. James’ uniform covering maps. Other results include:

- A morphism between inverse sequences of countable (possibly non-abelian) groups that induces isomorphisms on \lim and \lim^1 is invertible in the pro-category. This implies the “Whitehead theorem in Steenrod homotopy”, thereby answering two questions of A. Koyama.
- If X is an LC_{n-1} compactum, $n \geq 1$, its n -dimensional Steenrod homotopy classes are representable by maps $S^n \rightarrow X$, provided that X is simply connected. The assumption of simple connectedness cannot be dropped, by a well-known example of Dydak and Zdravkovska.
- A connected compactum is Steenrod connected (=pointed 1-movable) iff every its uniform covering space has countably many uniform connected components.

Contents

1. Introduction
 - A. Motivations
 - B. Basic notions
 - C. Steenrod theories versus singular ones
 - D. Comparison with literature
 - E. Synopsis
2. Steenrod homotopy category
3. Homotopy groups
4. Homology and cohomology
5. Some examples
6. Comparison of theories
7. Covering theory
8. Zero-dimensional homotopy

1. INTRODUCTION

Rephrasing a well-known quote of A. N. Whitehead, one can declare that the ultimate goal of Steenrod homotopy theory is to eliminate any need for general topology — by reducing it to combinatorics. In less figurative language, Steenrod homotopy theory is a natural framework for doing algebraic topology on quite general spaces in terms of algebraic topology of polyhedra. The principal purpose of this memoir is to provide a transparent treatment of Steenrod homotopy theory of *compacta*, i.e. compact metrizable spaces. An extension to residually finite-dimensional (=finitistic) metrizable complete uniform spaces is outlined at the end of §7. The exposition is essentially self-contained.

The specific content of the paper is summarized at the end of the introduction.

A. Motivations. The author’s interest in Steenrod homotopy was motivated by questions arising from other subjects.

1) D. Rolfsen’s 1972 problems on topological isotopy (=homotopy through embeddings) in \mathbb{R}^3 : Does there exist a knot that is not isotopic to the unknot? (Such a knot can only be wild, since every PL knot is isotopic to the unknot by a non-locally-flat PL isotopy.) Are isotopic PL links necessarily PL isotopic? An affirmative answer to the latter problem would follow from the analogue of the Vassiliev Conjecture (on completeness of finite type invariants of knots) for “links modulo knots” (i.e. PL links up to PL isotopy) [M1].

In general, it is natural to view Rolfsen’s problems as more intuitive counterparts of the Vassiliev Conjecture, which clarify the geometric meaning (see [M1]) of a \varprojlim -type obstruction to validity of the latter. Such an obstruction $\theta(f, g)$, defined for any pair of knots f, g that are indistinguishable by finite type invariants, arises from a reformulation of finite type invariants in terms of isovariant homotopy theory of appropriately compactified configuration spaces.¹ Besides, the author learned from different sources that some obstruction of this kind seems to have already been considered by R. Milgram in the 90s (unpublished). Incidentally, the knot theorist Ralph Fox must have had very similar ideas in mind when he switched to Steenrod homotopy in his last two papers (see §7).

¹Here is a sketch. If M is a smooth manifold, the configuration space $M^{(r)} := M^r \setminus (\text{all diagonals})$ has the “linear” Fulton–MacPherson compactification $M^{[r]}$ (see [Si]) and also a “polynomial” compactification $M^{\{r\}}$, which is obtained from $M^{[r]}$ by blowing up submanifolds of increasingly degenerate (as measured by the coranks of representative collections of vectors) limit configurations [M3]. Every PL or piecewise-smooth knot $f: S^1 \hookrightarrow S^3$ induces an S_r -equivariant stratum preserving map $f^{[r]}: (S^1)^{[r]} \rightarrow (S^3)^{[r]}$, and if f is smooth, $f^{[r]}$ is “aligned” in the sense of Sinha (see [BCSS]), or *equivalently*, lifts to an S_r -equivariant stratum preserving map $f^{\{r\}}: (S^1)^{\{r\}} \rightarrow (S^3)^{\{r\}}$, where $(S^1)^{\{r\}} = (S^1)^{[r]}$ since S^1 is one-dimensional. It now follows from the work of Sinha [Si] and Volić [V1], [V2] that smooth knots f and g are indistinguishable by rational Vassiliev invariants if $f^{\{r\}}$ and $g^{\{r\}}$ are homotopic in the space V_r of S_r -equivariant stratum preserving maps $(S^1)^{\{r\}} \rightarrow (S^3)^{\{r\}}$ for each r . (It is likely that the proof can be refined to work over the integers, cf. [Koyt]; to this one should add that it is still unknown whether general Vassiliev invariants reduce to rational ones.) Conversely, if f and g are indistinguishable by general Vassiliev invariants of types $< r$, by a result of Goussarov [Gu] and Habiro [Ha] they are related by smooth isotopy and C_r -moves. It is easy to see that each C_r -move looks just like an isotopy to any r -tuple of points on the knot; this means that $f^{\{r\}}$ and $g^{\{r\}}$ are related by a homotopy $h_r: I \rightarrow V_r$. Finally, if f and g are indistinguishable by all Vassiliev invariants, each h_r combines with $p_{r+1}h_{r+1}$, where $p_r: V_{r+1} \rightarrow V_r$ is the forgetful map, into a loop $\ell_r: (S^1, pt) \rightarrow (V_r, f^{\{r\}})$. Let $\theta(f, g)$ be the class of $([\ell_1], [\ell_2], \dots)$ in $\varprojlim \pi_1(V_r, f^{\{r\}})$. If f and g are related by a smooth isotopy φ_t , then each $p_{r+1}\varphi_t^{\{r+1\}} = \varphi_t^{\{r\}}$, whence $\theta(f, g)$ is trivial.

2) Borsuk's problem of embeddability of n -dimensional absolute retracts in \mathbb{R}^{2n} . By a classical result of Shchepin–Shtan'ko, embeddability of a given n -dimensional compactum X in \mathbb{R}^m in codimension $m - n \geq 3$ is equivalent to proper embeddability of an appropriate infinite mapping telescope $P_{[0,\infty)}$ (see §2) into $\mathbb{R}^m \times [0, \infty)$; see [MS] for a precise statement. In the metastable range $m > \frac{3(n+1)}{2}$, there is a complete obstruction to such an embeddability problem in equivariant stable cohomotopy, which reduces to an obstruction in ordinary cohomology when $m = 2n$ [MS]. In particular, it turns out that the impact of the LC_∞ condition for X on the embeddability of X in \mathbb{R}^{2n} is closely related to a certain subgroup in the Čech $2n$ -cohomology of $X \times X \setminus \Delta$, measuring the non-commutativity of \varprojlim (arising because $X \times X \setminus \Delta$ is non-compact) and \varinjlim (arising because X is not a polyhedron). This algebra suffices to construct a counterexample to the parametric version of the Borsuk problem, that is, an n -dimensional absolute retract, $n \geq 2$, admitting non-isotopic embeddings into \mathbb{R}^{2n+1} . The Borsuk problem itself turns out to be much harder, however.

3) R. D. Edwards' approach to the Hilbert–Smith Problem. Does there exist a free action of the group \mathbb{Z}_p of p -adic integers on an LC_∞ compactum with a finite dimensional orbit space? It is not hard to reformulate the property of X to be the orbit space of a free action of \mathbb{Z}_p on an LC_∞ compactum in terms of local Steenrod homotopy of X . This gives us a purely topological (with no group actions involved) — even combinatorial topological (due to the nature of Steenrod homotopy) problem to think about.

B. Basic notions. Let $X \subset \mathbb{R}^m$ be a compactum. The *Steenrod homotopy set* $\pi_n(X, x)$ is the set of equivalence classes of level-preserving maps

$$F: (S^n \times [0, \infty), b \times [0, \infty)) \rightarrow (\mathbb{R}^m \times [0, \infty), x \times [0, \infty))$$

such that the closure of $F(S^n \times [0, \infty))$ in $\mathbb{R}^m \times [0, \infty]$ meets $\mathbb{R}^m \times \{\infty\}$ in a subset of $X \times \{\infty\}$ — up to homotopy through maps of the same kind. As one would expect, $\pi_0(X, x)$ is a pointed set, π_1 is a group (using the base ray $b \times [0, \infty)$), and π_2, π_3, \dots are abelian groups.

Let $\hat{\pi}_n(X, x)$ stand for the *singular* homotopy set, i.e. the set of homotopy classes of maps $(S^n, b) \rightarrow (X, x)$. There is a homomorphism (pointed map when $n = 0$)

$$\hat{\pi}_n(X, x) \xrightarrow{\hat{\tau}} \pi_n(X, x)$$

assigning to a representing spheroid $f: (S^n, b) \rightarrow (X, x)$ the Steenrod spheroid $f \times \text{id}_{[0,\infty)}: S^n \times [0, \infty) \rightarrow X \times [0, \infty) \subset \mathbb{R}^m \times [0, \infty)$. It not hard to see that if X is a polyhedron, $\hat{\tau}$ is an isomorphism.

The definition of $\pi_n(X)$ can be formulated in a purely combinatorial way, without mention of the closure. Let us pick a sequence $\dots \subset P_1 \subset P_0$ of closed polyhedral neighborhoods of X in \mathbb{R}^m with $P_0 = \mathbb{R}^m$ and $\bigcap P_i = X$, and consider their mapping telescope $P_{[0,\infty)} = \bigcup P_i \times [i, i+1]$, which lies in $\mathbb{R}^m \times [0, \infty)$. Then $\pi_n(X, x)$ is simply the set of level-preserving homotopy classes of level-preserving maps

$$G: (S^n \times [0, \infty), b \times [0, \infty)) \rightarrow (P_{[0,\infty)}, x \times \{0, \infty\}).$$

(Indeed, every G is an F , and every F can be made into a G by an appropriate reparametrization of $[0, \infty)$, cf. the proof of Lemma 2.5.) This is a special case of

the definition in §3, which will not presuppose an embedding into \mathbb{R}^m nor that X is finite-dimensional.

Restricting a G as above to the integer levels of $S^n \times [0, \infty)$, we get a family of maps $g_i: (S^n, b) \rightarrow (P_i, x)$ such that each g_{i+1} is homotopic to g_i with values in P_i . This describes an epimorphism (pointed surjection when $n = 0$)

$$\pi_n(X, x) \xrightarrow{\tilde{\tau}} \varprojlim \pi_n(P_i, x).$$

The inverse limit² on the right hand side is also known as the “Čech homotopy group” $\tilde{\pi}_n(X, x)$. While being an invariant of (X, x) , it is of limited interest as such, mainly because it fails the homotopy exact sequence of a pair — which does hold for Steenrod homotopy groups. (The relative Steenrod and Čech homotopy groups are defined in the straightforward way.)

What is the kernel of $\tilde{\tau}$? An element of the kernel would be represented by a G whose restriction to every integer level is a constant map. Each $[i, i + 1]$ layer of G then gives rise to a map $G_i: (S^{n+1}, b) \rightarrow (P_i, x)$. The homotopy classes of these spheroids G_1, G_2, \dots are not entirely well defined, however. For we could subtract some piece (i.e., a spheroid) from some G_i and add it to G_{i+1} , without changing the class of G in $\pi_n(X)$. Limiting ourselves to the abelian situation ($n > 0$) for simplicity of notation, we conclude that $\ker \tilde{\tau}$ is precisely the cokernel of the homomorphism

$$\varphi: \prod_i \pi_{n+1}(P_i) \rightarrow \prod_i \pi_{n+1}(P_i)$$

given by $(\dots, g_1, g_0) \mapsto (\dots, g_1 - f_1(g_2), g_0 - f_0(g_1))$, where $f_i: \pi_{n+1}(P_{i+1}) \rightarrow \pi_n(P_i)$ are induced by the inclusions. This cokernel is known as the *derived limit* $\varprojlim^1 \pi_{n+1}(P_i)$ of the inverse sequence of our abelian groups. (Incidentally, note that the kernel of φ is nothing but $\varprojlim \pi_{n+1}(P_i)$.)

So, our findings can be summarized (for $n > 0$) by the short exact sequence

$$0 \rightarrow \varprojlim^1 \pi_{n+1}(P_i) \rightarrow \pi_n(X) \rightarrow \varprojlim \pi_n(P_i) \rightarrow 0. \quad (*)$$

Due to the straightforward nature of the inverse limit on the right, the derived limit on the left is undoubtedly the central algebraic object of Steenrod homotopy theory.

Let us now consider a (possibly unbounded) closed subset $X \subset \mathbb{R}^m$. Let $\dots \subset P_1 \subset P_0$ be closed polyhedral neighborhoods of X in \mathbb{R}^m with $P_0 = \mathbb{R}^m$ and $\bigcap P_i = X$ and such that each P_i is contained in the $\frac{1}{2^i}$ -neighborhood of X and contains the $\frac{1}{2^{i+1}}$ -neighborhood of X , with respect to the usual Euclidean metric

²The *inverse limit* $\varprojlim S_i$ of a sequence $\dots \xrightarrow{f_1} S_1 \xrightarrow{f_0} S_0$ of sets and maps is the subset of the product $\prod_i S_i$ consisting of all sequences (\dots, s_1, s_0) with $s_i = f_i(s_{i+1})$ for all i . Such sequences of points are called *threads*, and a sequence of sets and maps as above is called an *inverse sequence*; its maps may be referred to as the *bonding maps*. An inverse limit of groups (and homomorphisms) is naturally a group, and an inverse limit of topological spaces (and continuous maps) is naturally a topological space. For instance, X is homeomorphic to $\varprojlim (\dots \subset P_1 \subset P_0)$ in the above situation.

on \mathbb{R}^m . The *Steenrod homotopy set* $\pi_n(X, x)$ is again the set of level-preserving homotopy classes of level-preserving maps

$$G: (S^n \times [0, \infty), b \times [0, \infty)) \rightarrow (P_{[0, \infty)}, x \times \{0, \infty\})$$

into the mapping telescope $P_{[0, \infty)} = \bigcup P_i \times [i, i+1]$. If X is a uniform polyhedron (see [Is]), for instance, a union of cubes of the form $[i_1, j_1] \times \cdots \times [i_m, j_m]$, where $i_k, j_k \in \mathbb{Z}$ (possibly $i_k = j_k$), then the natural homomorphism $\hat{\pi}_n(X) \rightarrow \pi_n(X)$ is again an isomorphism. So in this case, $\pi_n(X)$ is a topological invariant of X . In general, it is not: if X is the pair of co-asymptotic hyperbolas $\{(x, y) \mid y = \pm \frac{1}{x}\}$ in the plane, then $\pi_0(X) = 0$. However, $\pi_n(X)$ is always a *uniform* invariant of X , i.e. it is invariant under homeomorphism that is uniformly continuous in both directions (see §7, where a more general definition of $\pi_n(X)$ is given, which does not presuppose an embedding into \mathbb{R}^m). In return, our $\pi_n(X)$ remains feasibly computable, as it still fits into the short exact sequence (*).

The said also applies to homology and cohomology. Recall that if K is a locally finite simplicial complex, in addition to the usual simplicial chains $C_*(K)$ and cochains $C^*(K) = \text{Hom}(C_*(K), \mathbb{Z})$ there are also the *compactly supported cochains* $C_c^*(K) = \varinjlim C^*(K, K_i)$, where the subcomplexes $\dots \subset K_1 \subset K_0$ have complements consisting of finitely many simplices, and $\bigcap K_i = \emptyset$,³ as well as the *locally-finite chains* $C_*^{\text{lf}}(K) = \varprojlim C_*(K, K_i) \simeq \text{Hom}(C_c^*(K), \mathbb{Z})$. The *locally-finite homology* $H_n^{\text{lf}}(P) = H_n(C_*^{\text{lf}}(K))$ of the locally compact polyhedron P , triangulated by K , and the *compactly supported cohomology* $H_{\text{lf}}^n(P) = H_{-n}(C_c^*(K))$ are invariants of *proper* homotopy equivalence. The Poincaré duality for possibly non-compact orientable m -manifolds reads: $H^n(M) \simeq H_{m-n}^{\text{lf}}(M)$, $H_n(M) \simeq H_c^{m-n}(M)$ (for manifolds with boundary there will be 4 different isomorphisms).

The groups H_c^n and H_n^{lf} can be very easily generalized to closed subsets of a Euclidean space, if we use the above notation. The *locally finite Steenrod homology* $H_n^{\text{lf}}(X)$ is defined to be $H_{n+1}^{\text{lf}}(P_{[0, \infty)})$, and the *Pontryagin cohomology with compact supports* $H_c^n(X)$ is $H_c^{n+1}(P_{[0, \infty)})$. It is not hard to show that although they are defined using the uniform neighborhoods P_i of X , these groups turn out to be topological (rather than just uniform) invariants of X . However, since they are not homotopy invariants of X (but only proper homotopy ones), their usage is limited.

The usual *Steenrod homology* $H_n(X)$ is defined to be the $(n+1)$ st homology group of the “[0, ∞)-locally finite” chain complex $C_*^{[0, \infty)}(P_{[0, \infty)}) = \varprojlim C_*(P_{[0, \infty)}, P_{[i, \infty)})$, where $P_{[0, \infty)}$ is triangulated so that each P_i is triangulated by a subcomplex, and we write P_J for the preimage of the subset $J \subset [0, \infty)$ under the projection $P_{[0, \infty)} \rightarrow [0, \infty)$. Similarly to the above, there is a natural short exact sequence

$$0 \rightarrow \varprojlim^1 H_{n+1}(P_i) \rightarrow H_n(X) \rightarrow \varprojlim H_n(P_i) \rightarrow 0. \quad (**)$$

This exact sequence (which may be identified as the universal coefficient formula for the left exact functor \varprojlim) is due to Milnor; by analogy, (*) is also known as the Milnor exact sequence.

³The *direct limit* $\varinjlim G_i$ of a sequence $G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} \dots$ of groups and homomorphisms is a group, whose elements are equivalence classes of *threads*, i.e. sequences of the form $g = (g_k, g_{k+1}, \dots)$, where each $g_{i+1} = f_i(g_i)$ for $i \geq k$. Threads g and $h = (h_l, h_{l+1}, \dots)$ are considered equivalent if $g_i = h_i$ for some $i \geq \max(k, l)$ (and hence also for $i+1, i+2, \dots$). The product of the classes of g and h is the class of $(g_m h_m, g_{m+1} h_{m+1}, \dots)$, where $m = \max(g, h)$.

The *Pontryagin cohomology* $H^n(X)$ is defined to be the $-(n+1)$ st homology group of the “[0, ∞)-compactly supported” cochain complex $C_{[0, \infty)}^*(P_{[0, \infty)}) = \varinjlim C^*(P_{[0, \infty)}, P_{[i, \infty)})$. Dually to (**), there is a natural isomorphism

$$H^n(X) \simeq \varinjlim H^n(P_i). \quad (**')$$

Instead of presenting a proof (it is easier than the above proof of (*) and (**)) and can be found in §4), let us clarify to some extent the content of the duality. If C_* is the chain complex

$$0 \leftarrow \bigoplus G_i \xleftarrow{\psi} \bigoplus G_i \leftarrow 0$$

where G_i are abelian groups and ψ sends every $g_i \in G_i$ to $g_i - f_i(g_i)$, then obviously $H_0(C_*) \simeq \varinjlim G_i$, $H^0(C_*) \simeq \varprojlim G_i^*$ and $H^1(C_*) \simeq \varprojlim^1 G_i^*$, where $G^* = \text{Hom}(G, \mathbb{Z})$. There is no surprise here: like any derived functor, \varprojlim^1 may be viewed as a topological object. We shall take advantage of this viewpoint in the proofs of Theorem 3.1(d) and Lemma 3.7(a).

C. Steenrod theories versus singular ones. A compactum X is said to be LC_∞ if it is locally n -connected for all n . For such compacta singular homotopy and homology coincide with Steenrod ones (see §6), and singular cohomology coincides with Pontryagin cohomology (see [Br]). For general compacta, however, it turns out that singular groups become somewhat “anomalous” and difficult to compute, whereas Steenrod groups still possess a number of reasonable properties which make them feasibly computable and applicable in geometric topology. Here are a few standard illustrations of this well-known thesis:

1) The Alexander duality $H^n(X) \simeq \tilde{H}_{m-n-1}(\mathbb{R}^m \setminus X)$ for compact X can use any kind of homology, but must use Pontryagin (not singular) cohomology. The Alexander duality $H_n(X) \simeq \tilde{H}^{m-n-1}(\mathbb{R}^m \setminus X)$ for compact X can use any kind of cohomology, but must use Steenrod (not singular) homology. See Theorem 4.3. Note that an immediate corollary of either of these isomorphisms is the Jordan curve theorem.

2) Finite-sheeted⁴ coverings over Steenrod connected (i.e. with trivial π_0) compacta are classified by the topologized Steenrod fundamental group π_1 (see Corollaries 7.5 and 7.8). They are not classified by the topologized singular fundamental group $\hat{\pi}_1$ (by considering the Warsaw circle, which is both Steenrod connected and path connected).

3) The group $[X, K(G, n)]$, defined either as the group of homotopy classes of maps $X \rightarrow K(G, n)$ or the group of Steenrod homotopy classes (see Proposition 2.3), is isomorphic to the Pontryagin (not singular) cohomology $H^n(X; G)$ for any countable abelian G (see Proposition 4.2).

4) Steenrod homology and Steenrod homotopy, not to mention Pontryagin cohomology, behave reasonably with respect to inverse limits (see (*), (**) and (**') above as well as Theorems 3.1(c) and 4.1(iii)), whereas singular homology and homotopy of simplest inverse limits is very difficult to compute (see Theorem 1.1 and Example 5.6).

⁴and, more generally, arbitrary coverings in the category of uniform spaces (rather than topological spaces)

5) Steenrod homology and Pontryagin cohomology satisfy a stronger form of excision and are characterized on pairs of compacta as being the only ordinary homology and cohomology theory satisfying this “map excision” axiom along with an additivity axiom [Mi] (see §4 below for the precise statement).

6) Pontryagin cohomology coincides with Alexander–Spanier cohomology [Sp] and with sheaf cohomology (with constant coefficients) [Br]. Steenrod homology coincides with Massey homology (which is the dual theory of Alexander–Spanier cohomology) [Mas] and, if the coefficients are finitely generated⁵, with the sheaf homology of Borel and Moore (with constant coefficients) [Br], [BoM]. Concerning all of these assertions see [Sk].

7) An n -dimensional compactum has trivial Steenrod homology and Pontryagin cohomology in dimensions $> n$; whereas singular homology tells nothing about dimension:

Theorem 1.1. (Barratt–Milnor [BM]) *Let E^n be the n -dimensional Hawaiian earring, that is the one-point compactification of $\mathbb{R}^n \times \mathbb{N}$. Then the singular $(2n-1)$ -homology $\hat{H}_{2n-1}(E^n) \neq 0$.*

The original proof was in the language of homotopy theory; below is a short proof based on the Pontryagin–Thom construction and linking numbers. It works to show that $\hat{H}_{2n-1}(E^n; \mathbb{Q})$ is uncountable. There seems to be no difficulty in reproving the entire result of [BM] — that $\hat{H}_{kn-k+1}(E^n; \mathbb{Q})$ is uncountable for each $k > 0$ — along the same lines by using higher Massey products.

Proof. Let $K_0, K_1, \dots \subset S^{2n-1}$ be a null-sequence of disjoint PL copies of S^{n-1} such that each K_{2i+1} is linked with K_{2i} with linking number $l_i > 0$ and unlinked with all other K_j . Let T_0, T_1, \dots be disjoint regular neighborhoods of the K_i ’s. Define a map $f: S^{2n-1} \rightarrow E^n$ by projecting each $T_i \cong S^{n-1} \times \mathbb{R}^n$ onto $\mathbb{R}^n \times \{i\}$ and sending the complement of $\bigcup T_i$ onto the compactifying point. We claim that $[f] \neq 0 \in \hat{H}_{2n-1}(E^n)$.

Suppose that $S^{2n-1} = \partial M$ for some compact oriented $2n$ -pseudo-manifold⁶ M such that f extends to a continuous map $F: M \rightarrow E^n$. Let us perturb F so as to make it simplicial with respect to some triangulations of $[-1, 1]^n \times \mathbb{N}$ and $F^{-1}([-1, 1]^n \times \mathbb{N})$ where each $x_i = (0, i) \in \mathbb{R}^n \times \mathbb{N}$ lies in the interior of an n -simplex. Then each $M_i := F^{-1}(x_i)$ is a compact oriented n -pseudo-manifold with boundary $\partial M_i = K_i$, whose regular neighborhood in M is homeomorphic to $M_i \times I^n$, see [Wi; Theorem 1.3.1]. Furthermore, M_i is co-oriented in M as long as an orientation of $\mathbb{R}^n \times \mathbb{N} \subset E$ is fixed. Let z_{2i} be the generator of $H^n(M, M \setminus M_{2i}) \simeq H^n(M_{2i} \times I^n, M_{2i} \times \partial I^n) \simeq H^0(M_{2i})$ corresponding to this co-orientation, and let c_{2i} be its image in $H^n(M)$. Suppose that some nontrivial linear combination $m_0 c_0 + \dots + m_r c_{2r} = 0 \in H^n(M)$. Let $R = M_0 \cup \dots \cup M_{2r}$. Then $m_0 z_0 + \dots +$

⁵If the coefficients are not finitely generated or not locally constant, the Borel–Moore theory is known to have certain “defects”, which are discussed in detail by E. G. Sklyarenko in the Editor’s comments to the Russian translation of the first edition of Bredon’s book [Br]. Instead, Sklyarenko proposes to construct a *cosheaf* Steenrod homology (not to be confused with the cosheaf Čech homology, defined in the second edition of Bredon’s book). Such a theory can indeed be constructed along the lines of Steenrod’s second definition of his homology (see §4 of the present paper); the details will appear elsewhere.

⁶We recall that a [pseudo-connected] (orientable) m -pseudo-manifold M with boundary ∂M is a polyhedral pair $(M, \partial M)$ admitting a triangulation $(K, \partial K)$ such that $[K \setminus K^{(m-2)}] \cup [\partial K \setminus (\partial K)^{(m-3)}]$ is a [connected] (orientable) m -manifold with boundary $\partial K \setminus (\partial K)^{(m-3)}$.

$m_r z_{2r} \in H^n(M, M \setminus R)$ is the image of some $b \in H^{n-1}(M \setminus R)$. The restriction of b over $X := \partial M \setminus (K_0 \cup \dots \cup K_{2r})$ equals $m_0 D[K_0] + \dots + m_r D[K_{2r}]$, where $D: H_{n-1}(K_0 \sqcup \dots \sqcup K_{2r}) \rightarrow H^{n-1}(X)$ is the Alexander duality. Hence $b(m_0[K_1] + \dots + m_r[K_{2r+1}]) = m_0^2 l_0 + \dots + m_r^2 l_r > 0$. On the other hand, each $[K_{2i+1}] = 0 \in H_{n-1}(M, M \setminus R)$ since $K_{2i+1} = \partial M_{2i+1}$. The obtained contradiction shows that c_{2i} are all linearly independent in $H^n(M)$. Thus $H^n(M)$ is not finitely generated, which is a contradiction. \square

D. Comparison with literature. It should be clarified (at the referee's request) that, to the best of the author's knowledge, the combinatorial treatment of Steenrod homotopy theory, as presented in this paper, has no "original exposition" in the literature, apart from certain fragments.⁷ At the same time, there is no lack in expositions of various closely related constructions, whose interrelations and relations with Steenrod homotopy are very confusing at the beginning; the basic relations are briefly summarized below. We shall not dwell on these technicalities in what follows; in particular, most of the "known results" mentioned in this paper are, strictly speaking, only reformulations of such, while their equivalence with the original formulations will sometimes be evident only to the experts.

The principal object of study in this paper is the Steenrod homotopy category, which in the case of compacta is defined in §2. In this case it coincides with the strong shape category, also known as the fine shape category [EH], [DS2], [KO] (see also [Fe2; §5], [C1], [IS] and a survey in [MaS; §III.9]). Moreover, in the case of compacta, the original shape category of Borsuk (see [F1], [DS1], [MaS]) has the same isomorphism classes of objects (see Proposition 2.7) and, at least in the finite dimensional case, the same isomorphisms (see Theorem 3.9); its morphisms are generally less informative (see examples in §5).

In general, the strong shape category has topological spaces as its objects and thus differs from the Steenrod homotopy category, whose objects are complete uniform spaces.⁸ There are many reasons for the latter; the most obvious one is that the homotopy exact sequence of a fibration fails for strong shape groups, already in the case of coverings over compacta.⁹ In view of the results of §7 this should not be surprising: one should rather expect to have a homotopy exact sequence of a uniform covering — which one indeed has in Steenrod homotopy. (The shape theorist might simply reply that not every infinite-sheeted covering over a compactum would then be considered a strong shape fibration; but for the geometric topologist, this only makes it more evident that strong shape theory cries out to be improved.) Another important reason is that a non-compact topological space

⁷These include Milnor's proof of Theorem 2 [Mi], Kodama's proof of Theorem 4 [Ko2] and the first paragraph of Ferry's proof of Theorem 4 [Fe2].

⁸This disagrees with the terminology of [EH] and [Po], whose "Steenrod homotopy" is only used as yet another synonym of strong shape.

⁹Let X be the Hawaiian snail in Example 5.7, that is the mapping torus of the self-homeomorphism $\sigma_n, (r, i) \mapsto (r, i + 1)$, of the n -dimensional Hawaiian earring $E^n = (\mathbb{R}^n \times \mathbb{Z})^+$, where $n \geq 2$. Then by the Milnor exact sequence $(*)$, $\pi_{n+1}(X) \simeq \mathbb{Z}[\mathbb{Z}]$ as a module over the group ring of the Steenrod fundamental group $\pi_1(X) \simeq \mathbb{Z}$. The universal cover \tilde{X} is (non-uniformly) homeomorphic to $E^n \times \mathbb{R}$, and therefore has the strong shape of E^n , for which $\pi_{n+1}(E^n) = 0$. Since E^n is compact, the $(n+1)$ st strong shape group of E^n is isomorphic to $\pi_{n+1}(E^n)$. Thus the $(n+1)$ st strong shape group of \tilde{X} vanishes and so the universal covering $\tilde{X} \rightarrow X$ does not induce an isomorphism of the $(n+1)$ st strong shape groups. At the same time, by (the non-compact case of) the same exact sequence $(*)$, the $(n+1)$ st Steenrod homotopy group $\pi_{n+1}(\tilde{X}) \simeq \mathbb{Z}[\mathbb{Z}]$; moreover, by the naturality of $(*)$, $\pi_{n+1}(\tilde{X}) \rightarrow \pi_{n+1}(X)$ is an isomorphism.

as simple as $[0, \infty)$ admits no cofinal sequence in the directed set of all coverings, and so the strong shape theory of metrizable topological spaces cannot do without non-sequential inverse spectra (indexed by directed sets other than the positive integers). Doing anything geometric with non-sequential mapping telescopes seems to be barely feasible; anyway it is not needed since the Steenrod homotopy category of metrizable complete uniform spaces manages well with inverse sequences.

Due to constraints of space and time, the extension of the Steenrod homotopy category to all separable metrizable complete uniform (SMCU) spaces has to be postponed to a separate paper, as it involves a solution to Isbell's problem [Is; Research Problem B₂] asking, essentially, whether infinite-dimensional uniform polyhedra can be defined in a meaningful way. Residually finite-dimensional SMCU spaces, i.e. inverse limits of finite-dimensional uniform polyhedra, are well understood after Isbell's work; the extension of the Steenrod homotopy category to such spaces is briefly outlined in §7 and is applied to Steenrod homotopy of compacta in §8. We could have as well defined the *uniform* Steenrod homotopy category for such spaces (whose restriction to uniform polyhedra is the uniform homotopy category, rather than the usual homotopy category) but we shall not need it. In [SSG], a peculiar *semi-uniform* Steenrod homotopy category has been defined for residually finite-dimensional SMCUs, based on the notion of a semi-uniform homotopy that is a (possibly non-uniformly-continuous) homotopy through uniformly continuous maps. We note that \mathbb{R} is semi-uniformly contractible, but not uniformly contractible, which should not be surprising since \mathbb{R} is not a uniform absolute retract (a notion whose definition does not involve any homotopies, cf. [Is]).

Steenrod homotopy and homology $\pi_i(X)$, $H_i(X)$ and Pontryagin cohomology $H^i(X)$, defined above in the case of closed subsets X of a Euclidean space, are invariants of Steenrod homotopy type (which in particular implies that they are well-defined as uniform invariants). The groups $H_i^{\text{lf}}(X)$ and $H_c^i(X)$ are invariants of proper strong shape (so in particular, of homeomorphism and of proper Steenrod homotopy type).

Steenrod homotopy groups of compacta were introduced by Christie [Ch] and rediscovered 30 years later by Quigley [Q1]. They are sometimes called “Quigley approaching groups” in the literature. Steenrod homology and Pontryagin cohomology of compacta have been introduced by Steenrod [St] and (in the finite-dimensional case) Pontryagin [P] themselves. Pontryagin also considered arbitrary closed subsets X of a Euclidean space and defined their Pontryagin cohomology with compact supports $H_c^i(X)$ — by means of equation (**'), where the cohomology of P_i 's is also taken to be with compact supports. Since these groups are topological invariants of X , Pontryagin did not have to worry about taking the neighborhoods P_i uniform in order to prove the well-definedness; but he was consistent in using only direct and inverse sequences (indexed by the natural numbers rather than general directed sets). Pontryagin cohomology $H^i(X)$ was considered in [Do] and [Miy] (see also [AgS]) under different names. As for Steenrod homology $H_i(X)$ (and homotopy $\pi_i(X)$) in the non-compact case, whose discussion Steenrod prudently avoided in the indicated paper, they might be not in the literature altogether. The locally finite Steenrod homology $H_i^{\text{lf}}(X)$ and Pontryagin cohomology with compact supports $H_c^i(X)$ were studied by Sklyarenko (see [Sk]) and others for arbitrary σ -compacta, i.e. locally compact separable metrizable spaces X .

Pontryagin's cohomology of compacta is better known as “Čech cohomology” in

the literature. Our primary concern is to avoid confusion of three different entities: (Alexandroff–)Čech(–Dowker) homology, which is not a homology theory as it does not satisfy the exactness axiom; Čech(–Dowker) cohomology, which is a cohomology theory of topological spaces; and Pontryagin cohomology, which is a cohomology theory of complete uniform spaces. In this paper we shall only need Pontryagin cohomology in the case of compacta, where it coincides with the Čech–Dowker cohomology. Even in this case it still seems worthwhile to call it “Pontryagin cohomology” as it helps eliminating the misleading (and historically unjustified!) link with the inexact Alexandroff–Čech homology.¹⁰

The literature gives the fullest account of two kinds of extension of Steenrod homology and Pontryagin cohomology of compacta to non-compact spaces: firstly, Steenrod–Sitnikov homology and the aforementioned Čech–Dowker cohomology, defined by Sitnikov and Dowker in the 50s using the direct limit (of a possibly non-sequential spectrum); and secondly, “strong” (or “coherent”) homology and cohomology, defined by Lisitsa–Mardešić and Miminoshvili in the 80s using homotopy inverse limit (of a possibly non-sequential spectrum). All these groups are strong shape invariants, as opposed to H_i and H^i , which are only Steenrod homotopy invariants. This means that H_i and H^i can detect finer phenomena (even though they are often easier to compute); specific examples are postponed to a subsequent paper by the author, which will also discuss interrelations of the aforementioned homology and cohomology theories.

E. Synopsis. We briefly review the content of the sections.

§2. *Steenrod homotopy category.* The section starts with our basic setup for the Steenrod theory and its justification, due to Milnor (2.1). Next we prove its equivalence with Christie’s original 1944 formulation (2.4) by a technique (2.5) that will be often used throughout the paper.

The leitmotif of further results of §2 is consideration of special situations that provide alternative viewpoints towards Steenrod homotopy as a whole. In brief: Steenrod homotopy vs. pro-homotopy (2.6, 2.8b) – Pointed vs. unpointed Steenrod homotopy (2.9) – Steenrod connected vs. path connected compacta (2.10) – Čech extension of functors (2.3) – Cell-like compacta (2.7). Overall, §2 aims to provide

¹⁰As for the history, it goes as follows. In his 1931 paper [P] in “Mathematische Annalen”, Pontryagin introduced direct limit of a direct sequence of groups, and used it to define cohomology of a finite-dimensional compactum [P; Ch. III, §II]. (He did not use the term “cohomology”, which only appeared several years later.) He also considered inverse sequences, but was able to define homology of a compactum only over the rationals (in Appendix II), proving incidentally that every inverse sequence of finite dimensional vector spaces over \mathbb{Q} is equivalent to an inverse sequence of epimorphisms. Čech was apparently unaware of this paper of Pontryagin when he wrote his 1932 paper [Č], which was his earliest on the subject. Anyway he did not attempt to consider cohomology or direct limits in that paper, but instead introduced *inverse* limit of an inverse spectrum of groups and used it to clarify the meaning of Alexandroff’s 1929 definition of homology of a compactum and to extend it to additional coefficient groups and to non-metrizable compact spaces. (Independent equivalent definitions of homology for metric compacta were also given by Vietoris (1927) and in the finite-dimensional case by Lefschetz (1930), though the equivalence of Alexandroff’s and Lefschetz’s approaches with that of Vietoris was only to be established much later [Le].) In the same paper Čech also attempted to define homology of non-compact spaces, but instead defined what in the locally compact case turns out to be the Alexandroff–Čech homology of the Stone–Čech compactification of the original space (cf. [ES; X.9.12]) — which is not a homotopy invariant of the original space. It was not until 1950 that Dowker “corrected” this definition of Čech.

a new simple approach to foundations of the theory; we also give a short proof of the Dydak–Segal “Fox theorem” in Steenrod homotopy (2.8a).

§3. *Homotopy groups.* Steenrod homotopy groups are defined, and basic tools of dealing with them are presented (3.1–3.4) and illustrated with the aid of UV_n compacta (3.5). The “exact continuity”¹¹ for Steenrod homotopy groups (3.1c) seems to be a new result. The main result of §3 is the “Whitehead theorem” in Steenrod homotopy (3.6), answering a 1983 question of A. Koyama. We also give two simple geometric proofs of the Edwards–Geoghegan stability theorem (3.10) and a simple proof of Dydak and Kodama’s theorem on stability of a union (3.14).

Finally, Steenrod fibrations are introduced and are found to be a simplified but equivalent version of the shape fibrations of Mardešić–Rushing. This enables us to simplify Cathey’s proofs of the exact sequence in Steenrod homotopy (3.15b) and Steenrod homotopy equivalence of fibers (3.15c) for Steenrod fibrations; and Mardešić and Rushing’s proof that cell-like maps between finite-dimensional compacta are Steenrod fibrations (3.15d). We also show that a Steenrod fibration over a connected compactum induces a surjection on π_0 (3.15b’).

§4. *Homology and cohomology.* We briefly recall the definition and simplest properties of Steenrod homology and Pontryagin cohomology groups (4.1, 4.2). As an illustration, we present an elementary proof of the Alexander duality theorems (4.3). A clean proof of the (absolute) “Hurewicz theorem” in Steenrod homotopy is given (4.4), along with a counterexample to a previously available “proof”.

§5. *Some examples.* With some exceptions, this and further sections presuppose only 2.1–2.5, 3.1–3.4, 4.1–4.2 and are independent of each other. §5 is a little zoo of compacta highlighting various features of Steenrod homotopy and homology as opposed to singular theories. These include the solenoid (5.3), the Hawaiian earring (5.6) and an alternative implementation of the Dydak and Zdravkovska example (5.8).

§6. *Comparison of theories.* This section is devoted to comparison of singular, Steenrod and Čech theories under various local connectivity hypotheses, as well as to comparison of these very hypotheses. The main result of §6 is a homotopy realization theorem (6.5). We also prove a parallel LC_{n-1} homology realization theorem (6.7), which is originally due to Shchepin (unpublished) and implies a 1999 result of Eda and Kawamura. Next, we give geometric proofs of Mardešić’s (6.8d) and Jussila’s (6.8b) comparison theorems and prove two further unpublished results by Shchepin: realization of Steenrod cycles by fractal pseudo-manifolds for HLC_{n-1} compacta (6.10) and the $HLC_n = HL\check{C}_n$ theorem (6.8a). On our way to these results, we reprove the homotopy comparison theorem of Hurewicz, Borsuk, et al. (6.1) and Ferry’s “Eventual Hurewicz theorem” (6.3).

As an application, we propose a geometric “explanation” of the finite generation of Steenrod homology of HLC_∞ compacta (6.11).

§7. *Covering theory.* We revisit Fox’s classification of overlayings, simplifying its statement and proof (7.4) and observe that overlayings coincide with uniform covering maps (7.6). The notion of universal generalized overlaying is illustrated with the “Nottingham compactum” (7.12).

¹¹f. the “Continuity versus Exactness” section in the Eilenberg–Steenrod book [ES]

§8. *Zero-dimensional homotopy.* This section is devoted to Steenrod connected (=pointed 1-movable) compacta. We obtain a simple proof of the Geoghegan–Krasinkiewicz theorem that virtually Steenrod disconnected compacta have empty Steenrod components (8.4). Brin’s counterexample (apparently unpublished) to its converse is apparently recovered (8.6). Next, we give a relatively simple proof of the Krasinkiewicz–Minc theorem (8.7), which implies, in particular, that path connected compacta are Steenrod connected. As a consequence, a characterization of Steenrod connected compacta in terms of the cardinality of uniform components of their uniform covering spaces is obtained (8.9). We also include simplified proofs of the McMillan–Krasinkiewicz theorem on continuous images (8.2) and Krasinkiewicz’s theorem on unions (8.3) of Steenrod connected compacta.

“Algebraic topology” content of the paper. A “classification” of sequential countable pro-groups in terms of \varprojlim and \varprojlim^1 is obtained (3.8), answering a question of Koyama. This is based on a lemma that “the vanishing of \varprojlim^1 implies the Mittag-Leffler condition for sequential countable pro-cosets” (3.7b). The paper includes simple expositions of known results about \varprojlim^1 and the Mittag-Leffler condition (3.1bd, 3.2, 3.3, 3.4, 3.7a, 3.11). There also is a simple geometric construction of J. H. C. Whitehead’s long exact sequence into which the Hurewicz homomorphisms in different dimensions fit (4.5).

“General topology” content of the paper. We introduce a notion of convergent inverse sequence of metrizable uniform spaces, which generalizes that of a Mittag-Leffler inverse sequence of sets (viewed as discrete uniform spaces), and presents an “explanation” of Bourbaki’s “Mittag-Leffler Theorem” (7.10) and consequently of the Baire Category Theorem (7.11).

Acknowledgements. This work would have never appeared without (i) a conversation with J. Dydak, who threw me into confusion by asking whether I ever gave a serious thought to the Hilbert–Smith Problem; (ii) S. P. Novikov’s 70th Anniversary, which caused me to begin writing a 5-page note on the subject of the Hilbert–Smith Problem, intended for a volume dedicated to that event; (iii) a correspondence with A. N. Dranishnikov, which helped me to realize that I needed to stop thinking about the Hilbert–Smith Problem until I had a better understanding of Steenrod homotopy; (iv) the prudence of the Editors of the aforementioned volume: V. M. Buchstaber, O. K. Sheinman and E. V. Shchepin — whose wisely calculated shifting of deadlines magically transformed the aforementioned short note into the present paper.

I’m grateful to the topology group of the University of Tennessee for their hospitality and encouragement during my visit there; and specifically to Jurek Dydak for very many helpful remarks and references to the literature and to Kolya Brodskiy, Pepe (José) Higes, Brendon LaBuz and Conrad Plaut for useful discussions and remarks on uniform spaces and uniform covering maps. Conversations with P. Akhmetiev, A. N. Dranishnikov, A. V. Chernavskij, R. Mikhailov, Yu. B. Rudyak, Yu. Turygin and E. V. Shchepin taking place over the years helped me to better understand Steenrod homology. I am particularly indebted to Misha Skopenkov for attentive listening that helped reveal two errors in time and for remarks on the text of the paper. In addition, I am grateful to N. Mazurenko, who helped translate the original English text into Russian.

2. STEENROD HOMOTOPY CATEGORY

Let X be a compactum. Then X is the inverse limit of an inverse sequence $P = (\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0)$ of compact polyhedra and PL maps (H. Freudenthal, 1937; a concise and clear exposition of the proof can be found in [Is; V.33]). The *mapping telescope* $P_{[0,\infty)}$ is the infinite union $\dots \cup_{P_2} MC(p_2) \cup_{P_1} MC(p_1)$ of the mapping cylinders of the bonding maps. It comes endowed with the projection $\pi_P: P_{[0,\infty)} \rightarrow [0, \infty)$, sending each $MC(p_i)$ onto $[i, i+1]$ in a level-preserving fashion. For each $J \subset [0, \infty)$, we denote $\pi_P^{-1}(J)$ by P_J . We shall assume that P_0 is a point; this does not restrict generality as we can always augment the given inverse sequence by the map $P_0 \rightarrow pt$ and shift the indices by one (these operations do not affect the inverse limit). This makes each $P_{[0,k]}$ contractible.

The inverse limit $P_{[0,\infty]}$ of the retractions $r_k: P_{[0,k+1]} \rightarrow P_{[0,k]}$ is easily seen to be a compactification of $P_{[0,\infty)}$ by X . The projections $\Pi_k: P_{[0,\infty]} \rightarrow P_{[0,k]}$ combine with the homotopies $r_k \simeq \text{id}_{P_{[0,k+1]}}$ into a homotopy $\Pi_t: P_{[0,\infty]} \rightarrow P_{[0,\infty]}$, $t \in [0, \infty]$, where Π_∞ is the identity. $P_{[0,\infty]}$ is LC_∞ since it ε -deformation retracts (via $\Pi_{1/\varepsilon}$) onto a compact polyhedron for each $\varepsilon > 0$.¹²

Let Y be another compactum, and let $Q = (\dots \xrightarrow{q_1^2} Q_1 \xrightarrow{q_0^1} Q_0)$ be an inverse sequence of compact polyhedra and PL maps ($Q_0 = pt$) with inverse limit Y .

Lemma 2.1. (a) (Milnor) *Every map $f: X \rightarrow Y$ extends to a continuous map $f_{[0,\infty]}: P_{[0,\infty]} \rightarrow Q_{[0,\infty]}$ that*

(i) *agrees with f on X and*

(ii) *restricts to a proper map $f_{[0,\infty)}: P_{[0,\infty)} \rightarrow Q_{[0,\infty)}$.*

(b) *Every two maps satisfying (i) and (ii) are homotopic through maps satisfying (i) and (ii).*

Part (a) is essentially [Mi; Theorem 2]. While Milnor's paper remained unpublished, the ANR $P_{[0,\infty]}$ and Lemma 2.1 were rediscovered in mid-70s [Ko1], [K1], [CS]; resp. [Ko2], [KOW] (see also [DS2; 3.4]). The “discrete telescope” $P_{\mathbb{N} \cup \infty} \subset P_{[0,\infty]}$ and the infinite mapping telescope $P_{[0,\infty)}$ were considered already by H. Freudenthal (cf. [K1]) and S. Lefschetz (cf. [Mi]) in 1930s. The following proof of Lemma 2.1 simplifies Krasinkiewicz's approach in [K1].

Proof. Let us triangulate $P_{[0,\infty)}$ so that the diameters of simplices tend to zero as they approach X , and let $r: X \cup P_{[0,\infty)}^{(0)} \rightarrow X$ be a retraction, where $P_{[0,\infty)}^{(0)}$ denotes the 0-skeleton of the triangulation. Let r_J be the restriction of r to $P_J^{(0)}$. Since q_i^∞ , q_i^{i+1} and r are uniformly continuous, for each i there exists a $j = j(i)$ such that the composition $P_{[j(i),\infty)}^{(0)} \xrightarrow{r_{[j(i),\infty)}} X \xrightarrow{q_i^\infty} Q_i$ extends “by linearity” to a map

$\varphi_i: P_{[j(i),\infty)} \rightarrow Q_i$, and moreover the composition $P_{[j(i+1),\infty)} \xrightarrow{\varphi_{i+1}} Q_{i+1} \xrightarrow{q_i^{i+1}} Q_i$ is homotopic “by linearity” to the restriction of φ_i .

Combining the appropriate restrictions of these maps and homotopies, we obtain an extension of f to a continuous map $F: P_{[k,\infty]} \rightarrow Q_{[0,\infty]}$, where $k = j(1)$, moreover $F(P_{[k,\infty)}) \subset Q_{[0,\infty)}$ by construction. The compact set $F(P_k)$ lies in some

¹²In fact, it is well-known (but not needed for our purposes) that $P_{[0,\infty]}$ is an ANR (and consequently an AR) by using either Lefschetz's or Hanner's characterization of ANRs (found respectively in Borsuk's and Hu's “Theories of Retracts”).

$Q_{[0,l]}$, which is contractible. Hence F extends to a map $f_{[0,\infty]}: P_{[0,\infty]} \rightarrow Q_{[0,\infty]}$ satisfying (i) and (ii). \square

(b). Since $P_{[0,\infty]}$ is LC_0 , the retraction r in the proof of (a) is homotopic to the identity keeping X fixed. Since $(P_{[0,\infty]}, X \cup P_{[0,\infty]}^{(0)})$ is a Borsuk pair (=the inclusion $X \cup P_{[0,\infty]}^{(0)} \hookrightarrow P_{[0,\infty]}$ is a cofibration), r extends to a continuous self-map $R: P_{[0,\infty]} \rightarrow P_{[0,\infty]}$, homotopic to the identity keeping X fixed. Then the given maps $F, G: P_{[0,\infty]} \rightarrow Q_{[0,\infty]}$ are homotopic to FR and GR keeping X fixed. Given an increasing sequence k_1, k_2, \dots , the telescopic bonding map $[q_0^{k_0}, q_\infty^{k_\infty}]: Q_{[k_0,\infty]} \rightarrow Q_{[0,\infty]}$ is homotopic to the identity keeping X fixed. Finally, $[q_0^{k_0}, q_\infty^{k_\infty}]FR$ is homotopic rel X to $[q_0^{k_0}, q_\infty^{k_\infty}]GR$ for an appropriate sequence (k_i) by the proof of part (a), applied to $f \times \text{id}_I: X \times I \rightarrow Y$. \square

Corollary 2.2. *When $Y = X$, $(\text{id}_X)_{[0,\infty]}: P_{[0,\infty]} \rightarrow Q_{[0,\infty]}$ is a proper homotopy equivalence.*

Steenrod homotopy category. We define a *Steenrod homotopy class* $X \rightsquigarrow Y$ to be the proper homotopy class of a proper map $f: P_{[0,\infty]} \rightarrow Q_{[0,\infty]}$. By virtue of the proper homotopy equivalences $(\text{id}_X)_{[0,\infty]}$ and $(\text{id}_Y)_{[0,\infty]}$, this definition does not depend on the choice of the inverse sequences P and Q . The composition $X \rightsquigarrow Y \rightsquigarrow Z$ is defined by composing appropriate proper maps. Assigning to a map $f: X \rightarrow Y$ the proper homotopy class, denoted $[f]$, of $f_{[0,\infty]}$, yields a map $\hat{\tau}$ from the set $[X, Y]^\wedge$ of homotopy classes of maps $X \rightarrow Y$ into the set $[X, Y]$ of Steenrod homotopy classes $X \rightsquigarrow Y$, which is natural in both variables. A Steenrod homotopy class $f: X \rightsquigarrow Y$ is a *Steenrod homotopy equivalence* if there exists a $g: Y \rightsquigarrow X$ such that $gf = [\text{id}_X]$ and $fg = [\text{id}_Y]$; in other words, if f is represented by a proper homotopy equivalence. When such an f exists, X and Y are said to be of the same *shape* (Steenrod homotopy type).

Steenrod homotopy classes were introduced (under a different name) by D. E. Christie in his 1944 dissertation [Ch]. He gave several equivalent definitions and found uncountably many Steenrod homotopy classes from a point into the dyadic solenoid (concerning these see Example 5.3). Christie's versions of the definition (see Proposition 2.4(a) below) as well as Ferry's version [Fe2; §5] (see Proposition 2.4(b)) are slightly simpler in that they partially bypass Lemma 2.1, but on the other hand their asymmetry does not allow to define composition of Steenrod homotopy classes. It appears that Christie himself was unaware of Steenrod's 1940 paper [St], where Steenrod homology was introduced (and, among other things, uncountably many 0-dimensional Steenrod homology classes of the dyadic solenoid were found). However, the paper of Steenrod is found in the reference list of the 1942 book [Le] by Christie's thesis advisor Lefschetz.

On several occasions (in 3.10, 4.2, 7.4) we will need Steenrod homotopy classes from a compactum to a possibly non-compact polyhedron, such as the classifying space¹³ of a countable group. If $P = (\dots \rightarrow P_1 \rightarrow P_0)$ and $Q = (\dots \rightarrow Q_1 \rightarrow Q_0)$ are inverse sequences of (possibly non-compact) polyhedra, we call a map $Q_{[0,\infty]} \rightarrow R_{[0,\infty]}$ between the two infinite mapping telescopes *semi-proper* if for each k there exists an l such that $f^{-1}(R_{[0,k]}) \subset Q_{[0,l]}$. Now if all the P_i are

¹³Recall that every countable CW-complex is homotopy equivalent to a locally finite polyhedron. For instance, the homotopy type of $K(\mathbb{Z}/2, 1)$ is represented by the direct mapping telescope of the sequence of inclusions $\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \dots$.

compact and $X = \varprojlim P_i$; and $Q_0 = pt$ and $Q_i = K$ for all $i > 0$, where K is a non-compact polyhedron, we define a *Steenrod homotopy class* $X \rightsquigarrow K$ to be the semi-proper homotopy class of a semi-proper map $F: P_{[0,\infty)} \rightarrow Q_{[0,\infty)}$. An equivalent definition will be given in §7 in the case where K is finite-dimensional. We write $[X, K]$ for the set of Steenrod homotopy classes $X \rightsquigarrow K$. Lemma 2.1 yields a map $\hat{\tau}: [X, K]^\Delta \rightarrow [X, K]$.

Proposition 2.3. (a) *If K is a polyhedron, $\hat{\tau}: [X, K]^\Delta \rightarrow [X, K]$ is a bijection.*

(b) *If K is a polyhedron and X is the inverse limit of polyhedra P_i , then there is a natural bijection $[X, K] \rightarrow \varinjlim [P_i, K]$.*

Part (a) is found, for instance, in [DS2; 4.4], and (b) is also well-known.

Proof. (a) Since K is a polyhedron, $Q_{[0,\infty)}$ can be chosen to be the open cone $CK \cup_{K=K \times 0} K \times [0, \infty)$. Every semi-proper map of $P_{[0,\infty)}$ to this open cone restricts to a map $P_{[k,\infty)} \rightarrow K \times [0, \infty) \simeq K$ for some k . On the other hand, every $f: P_{[k,\infty)} \rightarrow K$ is homotopic to an f' satisfying $f'|_{P_{i+1}} = (f'|_{P_i})p_i$ for each i , and therefore extending by continuity over X . By repeating the same construction for homotopies, we obtain a composition $[X, K] \xrightarrow{\varphi} \varinjlim S_i \xrightarrow{\psi} \varinjlim [P_{[k,\infty)}, K] \xrightarrow{\chi} [X, K]^\Delta$, where S_k is the set of semi-proper homotopy classes of semi-proper maps $P_{[k,\infty)} \rightarrow K \times [0, \infty)$. On the other hand, $\hat{\tau}$ factors as $[X, K]^\Delta \xrightarrow{\chi'} \varinjlim [P_{[k,\infty)}, K \times [0, \infty)] \xrightarrow{\psi'} \varinjlim S_k \xrightarrow{\varphi'} [X, K]$ by construction. It is easy to check that $\varphi' = \varphi^{-1}$, $\psi' = \psi^{-1}$ and $\chi' = \chi^{-1}$. In particular, to see that $\chi'\chi = \text{id}$, note that if two maps $P_{[k,\infty)} \rightarrow K$ agree on X , their restrictions to some $P_{[l,\infty)}$ are homotopic. \square

(b). Each P_i is a deformation retract of $P_{[i,\infty)}$, so $\varinjlim [P_i, K] = \varinjlim [P_{[i,\infty)}, K]$. The latter coincides with $[X, K]$ by the proof of (a). \square

Let us retreat to the case of Steenrod homotopy classes from a compactum to a compactum.

Proposition 2.4. *Let Y be the limit of an inverse sequence of compact polyhedra Q_i , where Q_0 is not assumed to be a point. Steenrod homotopy classes $X \rightsquigarrow Y$ correspond bijectively to:*

- (a) *classes of level preserving maps $X \times [0, \infty) \rightarrow Q_{[0,\infty)}$ up to level preserving homotopy;*
- (b) *proper homotopy classes of proper maps $X \times [0, \infty) \rightarrow Q_{[0,\infty)}$.*

Note that the (most useful) case where X is a polyhedron is obvious, modulo Lemma 2.5(a) below, since $P_{[0,\infty)}$ can be chosen to be the open cone $CX \cup_{X=X \times \{0\}} X \times [0, \infty)$ and then CX can be discarded along with $Q_{[-1,0]} := CQ_0$.

The statement of part (b) is implicit in the definitions of [Fe2; §5], and it might be possible to extract a proof from [DS2] (see in particular Lemma 3.4 there).

Proof. If $Q_0 = pt$, a Steenrod homotopy class immediately yields a proper homotopy class as in (b), and by Lemma 2.5(a) below, a proper homotopy class as in (b) yields a level-preserving homotopy class as in (a). The general case reduces to the case $Q_0 = pt$ similarly to Lemma 2.5(a).

Let $f_{[0,\infty)}: X \times [0, \infty) \rightarrow Q_{[0,\infty)}$ be level-preserving. It follows from Lemma 2.1(a) that each $f_i: X \rightarrow Q_i$ factors through a map $F_i: P_{[k_i,\infty)} \rightarrow Q_i$ for some k_i .

From Lemma 2.1(b), the composition $P_{[k_{i+1}, \infty]} \xrightarrow{F_{i+1}} Q_{i+1} \xrightarrow{q_i^{i+1}} Q_i$ restricted to $P_{[l_i, \infty]}$ for some $l_i \leq \min(k_i, k_{i+1})$ is homotopic to the restriction of F_i . Without loss of generality, the sequence $l = (l_1, l_2, \dots)$ is non-decreasing. Then $f_{[0, \infty)}$ factors through a level-preserving map $R_{[0, \infty)} \rightarrow Q_{[0, \infty)}$, where $R_i = P_{[l_i, \infty]}$ with bonding maps $R_{i+1} \rightarrow R_i$ being the inclusions. Then $f_{[0, \infty)}$ also factors through a level-preserving map $f'_{[0, \infty)}: P_{[0, \infty)}^l \rightarrow Q_{[0, \infty)}$, where $P_i^l = P_{l_i}$.

A similar argument shows that if $f_{[0, \infty)}$ is level-preserving homotopic to $g_{[0, \infty)}$, then $f'_{[0, \infty)}$ and the similarly constructed $g': P_{[0, \infty)}^m \rightarrow Q_{[0, \infty)}$ are level-preserving homotopic when precomposed with the level-preserving maps $P_{[0, \infty)}^n \rightarrow P_{[0, \infty)}^l$ and $P_{[0, \infty)}^n \rightarrow P_{[0, \infty)}^m$ for some sequence $n = (n_i)$ majorizing both l and m . \square

Lemma 2.5. (a) *Every proper map $X \times [0, \infty) \rightarrow Q_{[0, \infty)}$ is proper homotopic to a level-preserving map. Proper homotopic level-preserving maps $X \times [0, \infty) \rightarrow Q_{[0, \infty)}$ are level-preserving homotopic.*

(b₀) *Every proper map $f: P_{[0, \infty)} \rightarrow Q_{[0, \infty)}$ is properly homotopic to an f' such that for some infinite subsequence $P_i^k = P_{k_i}$, the composition of the “reindexing” proper homotopy equivalence $k_{[0, \infty)}: P_{[0, \infty)}^k \rightarrow P_{[0, \infty)}$ and f' is level-preserving.*

(b₁) *If level-preserving maps $f, g: P_{[0, \infty)} \rightarrow Q_{[0, \infty)}$ are properly homotopic, there exists an infinite increasing sequence $k = (k_i)$ such that the compositions of $f^k, g^k: P_{[0, \infty)}^k \rightarrow Q_{[0, \infty)}^k$ with the level-preserving map $[p_0^{k_0}, p_\infty^{k_\infty}): Q_{[0, \infty)}^k \rightarrow Q_{[0, \infty)}$ are level-preserving homotopic.*

Parts (b₀) and (b₁) seem to be parallel to [EH; 3.7.20].

Proof. (a). If $\Phi: X \times [0, \infty) \rightarrow Q_{[0, \infty)}$ is proper, $\varphi: [0, \infty) \rightarrow [0, \infty)$ defined by $\varphi(t) = \inf(p(\Phi(X \times [t, \infty))))$ is a proper nondecreasing continuous function, so can be approximated by a homeomorphism $\tilde{\varphi}$ such that $\tilde{\varphi}(t) \leq \varphi(t)$ for every t . Then Φ is properly homotopic to Φ' defined by $\Phi'(x, t) = \Pi_{\tilde{\varphi}(t)}(\Phi(x, t))$. Finally, Φ' is homotopic to Φ'' defined by $\Phi''(x, t) = \Phi'(x, \tilde{\varphi}^{-1}(t))$, which is level-preserving.

To prove the second assertion it suffices to observe that the homotopy from Φ to Φ'' may be assumed to depend continuously on Φ and to be the identity whenever Φ is level-preserving. \square

(b). This is similar to (a). The proper map $k_{[0, \infty)}$ replaces the passage from Φ' to Φ'' , and the level-preserving map $[p_0^{k_0}, p_\infty^{k_\infty})$ arises on passing from Φ to Φ' . \square

Proposition 2.6. *Let X and Y be the limits of inverse sequences of compact polyhedra $P = (\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0)$ and $Q = (\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0)$. The following are equivalent:*

- (i) *X and Y are of the same shape;*
- (ii) *$P \simeq Q$ in pro-homotopy, that is, there exist an increasing sequence $k: \mathbb{N} \rightarrow \mathbb{N}$ and level-preserving maps $f_{[0, \infty)}: P_{[0, \infty)}^k \rightarrow Q_{[0, \infty)}^k$ and $g_{[0, \infty)}: Q_{[0, \infty)}^{k_s} \rightarrow P_{[0, \infty)}^k$, where P_i^k stands for $P_{k(i)}$ and $s(i) = i + 1$, such that $g_i f_{i+1}: P_{k(i+1)} \rightarrow P_{k(i)}$ and $f_i g_i: Q_{k(i+1)} \rightarrow Q_{k(i)}$ are homotopic to the bonding maps for all i ;*
- (iii) *P and Q are equivalent under the equivalence relation generated by passing to subsequences and homotoping the bonding maps.*
- (iv) *$P_{[0, \infty)}$ and $Q_{[0, \infty)}$ have the same proper simple homotopy type.*

The proof that (ii) \Rightarrow (i) traces back to [EH; 5.2.9]. The equivalence relation of (iii) was introduced by Siebenmann [S2].

Proof. (iii) \Rightarrow (iv) is straightforward (see details in [S2]), and (iv) \Rightarrow (i) is immediate from definitions. (i) \Rightarrow (ii) follows using Lemma 2.5(b). Finally, assuming (ii), both P and Q are equivalent in the sense of (iii) to the combined inverse sequence $\dots \xrightarrow{g_1} P_{k(1)} \xrightarrow{f_1} Q_{k(1)} \xrightarrow{g_0} P_{k(0)} \xrightarrow{f_0} Q_{k(0)}$. \square

Proposition 2.7. (Lacher [La]) *A compactum X has the shape of a point iff it is the limit of an inverse sequence of maps between PL balls.*

Compacta that satisfy any of the equivalent conditions of Proposition 2.7 are called *cell-like* in the literature. Maps with cell-like point-inverses will be discussed in Theorem 3.15(d,e).

Proof. Suppose that X is the inverse limit of polyhedra P_i . To prove the “if” direction, it suffices to note that if P_i is contractible, $f_i: P_{i+1} \rightarrow P_i$ is the bonding map and h_i is a null-homotopy of id_{P_i} , the compositions $h_i f_i$ and $f_i h_{i+1}$ are homotopic.

Conversely, suppose that X has the shape of a point. Then by Proposition 2.6, for each i there exists a $j = j(i) > i$ such that the bonding map $P_j \rightarrow P_i$ is null-homotopic. Then it factors through the inclusion of P_j into the cone CP_j . Let $j_0 = 0$ and $j_{k+1} = j(j_k)$, and set $Q_k = CP_{j_k}$. Then X is the inverse limit of the maps $Q_{k+1} \rightarrow P_{j_k} \subset Q_k$. Each cone Q_k is collapsible, so a regular neighborhood of a copy of it in some Euclidean space is a PL ball B^{n_k} . Then X is the inverse limit of the maps $B^{n_{k+1}} \rightarrow Q_{k+1} \rightarrow Q_k \subset B^{n_k}$. \square

Note that Proposition 2.6 does not assert that $f: X \rightsquigarrow Y$ is a Steenrod homotopy equivalence iff it is represented by an isomorphism in pro-homotopy. This is known to be true in the finite dimensional case (see Theorem 3.9); the general case remains open (see [DS2] and [Gü] for partial results). Another result in this direction is the following version “for maps” of Proposition 2.6.

Proposition 2.8. *Let X and Y be compacta.*

(a) (Dydak–Segal) *Every Steenrod homotopy class $f: X \rightsquigarrow Y$ factors into the composition $h[i]$ of an embedding $i: X \hookrightarrow Z$ and a Steenrod strong deformation retraction (so, in particular, a Steenrod homotopy equivalence) $h: Z \rightsquigarrow Y$.*

(b) *Let $f: X \hookrightarrow Y$ be an inclusion map. The following are equivalent:*

- (i) *f is a Steenrod homotopy equivalence;*
- (ii) *X is a Steenrod strong deformation retract of Y ;*
- (iii) *(X, Y) is an inverse limit of compact polyhedral pairs (Q_i, P_i) such that, writing $f_{[0,\infty)}: P_{[0,\infty)} \rightarrow Q_{[0,\infty)}$ for the inclusion map, there exists a level-preserving map $g_{[0,\infty)}: Q_{[0,\infty)}^s \rightarrow P_{[0,\infty)}$, where $Q_i^s = Q_{i+1}$, such that each $g_i f_{i+1}: P_{i+1} \rightarrow P_i$ equals the bonding map, and each $f_i g_i: Q_{i+1} \rightarrow Q_i$ is homotopic $\text{rel } P_{i+1}$ to the bonding map.*

A Steenrod homotopy class $h: Z \rightsquigarrow Y$ is a *Steenrod strong deformation retraction* if $Y \subset Z$, and the pair (Z, Y) is an inverse limit of compact polyhedral pairs (Q_i, P_i) such that h is represented by a proper strong deformation retraction $Q_{[0,\infty)} \rightarrow P_{[0,\infty)}$.

Part (a) was proved in [DS2] by a different construction, occupying pp. 24–26 there. The implication (i) \Rightarrow (ii) is related to [C1; 1.15] (see also [DS2; 4.5, 6.3']). The idea of the “algebraic” proof of (iii) \Rightarrow (i) goes back to [DS2; 6.2], and the construction used in the final part of the “geometric” proof is found in [EH; proof of 3.7.4].

Proof. (a). By Lemma 2.5(b), f can be represented by a level-preserving map $f_{[0,\infty)}: P_{[0,\infty)} \rightarrow Q_{[0,\infty)}$, where $\varprojlim P_i = X$ and $\varprojlim Q_i = Y$. Every bonding map $P_{i+1} \rightarrow P_i$ extends to a map $MC(f_{i+1}) \rightarrow MC(f_i) \cup Q_{[i,i+1]} \rightarrow MC(f_i)$ between the mapping cylinders, which also restricts to the bonding map $Q_{i+1} \rightarrow Q_i$. Writing $R_i = MC(f_i)$, clearly $R_{[0,\infty)}$ proper strong deformation retracts onto $Q_{[0,\infty)}$. The retraction $h_{[0,\infty)}: R_{[0,\infty)} \rightarrow Q_{[0,\infty)}$ represents a Steenrod strong deformation retraction $h: Z \rightsquigarrow Y$, where $Z = \varprojlim R_i$. Clearly, $f_{[0,\infty)}$ is properly homotopic to the composition of the inclusion map $P_{[0,\infty)} \hookrightarrow R_{[0,\infty)}$ with $h_{[0,\infty)}$. \square

(b). (i) \Rightarrow (ii) \Rightarrow (iii). Let us represent f by a proper map $f_{[0,\infty)}: P_{[0,\infty)} \rightarrow Q_{[0,\infty)}$. Assuming (i), $f_{[0,\infty)}$ is a proper homotopy equivalence. Then by the proofs of standard results of homotopy extension theory [Sp; 1.4.10, 1.4.11, 1.D.2], there exists a proper strong deformation retraction of $Q_{[0,\infty)}$ onto $P_{[0,\infty)}$. This proves (ii). The implication (ii) \Rightarrow (iii) follows using Lemma 2.5(b). \square

(iii) \Rightarrow (i). *Algebraic proof.* Writing $(Q, P) = (\cdots \rightarrow (Q_1, P_1) \rightarrow (Q_0, P_0))$, the hypothesis implies that (Q, P) is isomorphic to (P, P) in pro-homotopy. Then by the proof of Proposition 2.6, (Y, X) has the shape of (X, X) . If $\varphi: (Y, X) \rightsquigarrow (X, X)$ and $\psi: (X, X) \rightsquigarrow (Y, X)$ are such that $\varphi\psi = [\text{id}_{(X,X)}]$ and $\psi\varphi = [\text{id}_{(Y,X)}]$, then $\psi: X \rightsquigarrow Y$, $\varphi: X \rightsquigarrow X$ and the inclusion $i: Y \rightarrow X$ satisfy $\psi\varphi[i] = [\text{id}_Y]$ and $[i]\psi\varphi = [\text{id}_X]$. Thus $[i]$ is a Steenrod homotopy equivalence. \square

Geometric proof. Since $g_i|_{P_{i+1}}$ equals the bond, the mapping cylinder $MC(g_i)$ contains $P_{[i,i+1]}$. The homotopy between $f_i g_i$ and the bonding map yields a homotopy equivalence $\varphi: MC(g_i) \rightarrow Q_{[i,i+1]}$ restricting to the identity on $Q_i \cup P_{[i,i+1]}$. Now $Q'_i := MC(g_i)$ and $P'_i := P_{[i,i+1]}$ can be arranged into inverse sequences with bonding maps $Q'_{i+1} \xrightarrow{\pi\varphi} Q_{i+1} \subset Q'_i$ and $P'_{i+1} \xrightarrow{\pi} P_{i+1} \subset P'_i$, where each π projects the mapping cylinder onto its target space. Since $\cdots \rightarrow (Q'_1, P'_1) \rightarrow (Q'_0, P'_0)$ is equivalent to $\cdots \rightarrow (Q_1, P_1) \rightarrow (Q_0, P_0)$ under the equivalence relation of Proposition 2.6(iii), the inclusion map $f'_{[0,\infty)}: P'_{[0,\infty)} \rightarrow Q'_{[0,\infty)}$ represents the Steenrod homotopy class of f . On the other hand, each Q'_i obviously collapses onto P'_i .

The strong deformation retraction of Q'_i onto P'_i yields a deformation retraction of $Q'_i \times I$ onto $P'_i \times I \cup Q'_i \times \partial I$. Similarly, $Q'_{[i,i+1]}$ deformation retracts onto $P'_{[i,i+1]} \cup Q'_{i+1} \cup Q'_i$. Composing the latter retractions with the retractions $Q'_i \rightarrow P'_i$, we obtain a proper deformation retraction $r: Q'_{[0,\infty)} \rightarrow P'_{[0,\infty)} \cup Q'_{\mathbb{N}} \rightarrow P'_{[0,\infty)}$. Thus r is a proper homotopy equivalence inverting the proper homotopy class of the inclusion $f'_{[0,\infty)}$. \square

A map $f: X \rightarrow Y$ (respectively, a Steenrod homotopy class $[F]: X \rightsquigarrow Y$) between compacta with basepoints $x \in X$, $y \in Y$ is called *pointed* if it is basepoint preserving, i.e. $f(x) = y$ (respectively, if the proper map $F: P_{[0,\infty)} \rightarrow Q_{[0,\infty)}$ is base ray preserving, i.e. $F(p_{[0,\infty)}) \subset q_{[0,\infty)}$, where p_i and q_i are the images of x and y in P_i and Q_i). The set $[(X, x), (Y, y)]$ of pointed Steenrod homotopy classes receives the natural structure of a pointed set (i.e. a set with a distinguished point).

Proposition 2.9. *Let X and Y be connected compacta with basepoints x and y , and let $f: (X, x) \rightsquigarrow (Y, y)$ be a Steenrod homotopy class. If the underlying unpointed class $\bar{f}: X \rightsquigarrow Y$ of f is a Steenrod homotopy equivalence then f is a Steenrod homotopy equivalence.*

This was proved in [DS2; 4.6] and [C1; 1.16 (see also 1.14, 1.15)].

Proof. By the pointed version of Proposition 2.8(a), we may assume without loss of generality that f is an inclusion map (then, in particular, $x = y$). Then by (i) \Rightarrow (ii) in Proposition 2.8(b), X is a Steenrod strong deformation retract of Y . But this implies that (X, x) is a Steenrod strong deformation retract of (Y, y) . \square

A compactum X is *Steenrod connected* if there is only one Steenrod homotopy class from a point into X . Proposition 2.9 implies that there is no difference between pointed and unpointed shape for Steenrod connected compacta.

Example 2.10 (topologist's sine curve). Let $X \subset [0, 1] \times [-1, 1]$ be the closure of the graph of $\sin \frac{1}{x}$, where x runs over $(0, 1]$. Then X is Steenrod connected (in fact, even cell-like), but not path connected.

Steenrod connected compacta will be considered in more detail in §8. For instance, Theorem 8.7 (due originally to Krasinkiewicz and Minc) implies that a compactum X is Steenrod connected if and only if every two maps $pt \rightarrow X$ represent the same Steenrod homotopy class. In particular, path connected compacta are Steenrod connected.

On the other hand, as observed in [GK; Remark 10.1], the set of Steenrod homotopy classes $pt \rightsquigarrow X$ that are representable by maps is not a shape invariant: if $\alpha: pt \rightsquigarrow X$ is not representable by a map, represent it by a level-preserving map $f: [0, \infty) \rightarrow P_{[0, \infty)}$, where $X = \varprojlim P_i$. Then $Y := X \cup f([0, \infty))$ is Steenrod homotopy equivalent to X , but $\alpha: pt \rightsquigarrow Y$ is now represented by $f|_{\{0\}}$.

3. HOMOTOPY GROUPS

The definition of the set $[X, Y]$ of Steenrod homotopy classes $X \rightsquigarrow Y$ generalizes to pairs of compacta in the obvious way, based on the fact that a pair of compacta is an inverse limit of pairs of compact polyhedra. We define the *Steenrod homotopy set* $\pi_n(X; x) = [(S^n, pt), (X, x)]$. Thus (see Proposition 2.4(a)), $\pi_n(X; x)$ can be identified with the set of level-preserving homotopy classes of level-preserving maps

$$(S^n \times [0, \infty), pt \times [0, \infty)) \rightarrow (P_{[0, \infty)}, p_{[0, \infty)})$$

where $P = (\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0)$ is an inverse sequence of compact polyhedra and PL maps with inverse limit X ; and $p_{[0, \infty)}$ is the mapping telescope of the images p_i of the basepoint x in the P_i 's. By Proposition 2.3(a), if X is a polyhedron, $\hat{\tau}: \hat{\pi}_n(X; x) \rightarrow \pi_n(X; x)$ is a bijection.

The group operations (for $n \geq 1$) and the induced maps f_* , where f is a pointed (i.e. basepoint preserving) map or a pointed (i.e. base ray preserving) Steenrod homotopy class, are defined in the obvious way. We suppress the basepoint from notation and write $\pi_n(X)$ when it does not lead to a confusion. Next, let $\pi_n(X, A; x) = [(D^n, \partial D^n, pt), (X, A, x)]$ for any pair of compacta (X, A) with a basepoint $x \in A$ and any $n \geq 1$. More generally, we allow any Steenrod homotopy class $x: pt \rightsquigarrow A$ as a basepoint. We also write $\pi_n(X, A)$ when the basepoint is not explicitly used. The group operations on $\pi_n(X, A)$ are defined in the obvious way for $n \geq 2$. There is an action of $\pi_1(A)$ on $\pi_n(X)$ for $n \geq 1$ and on $\pi_n(X, A)$ for $n \geq 2$, and an action of $\pi_1(X)$ on $\pi_1(X, A)$. We leave $\pi_0(X, A)$ undefined.

Topology on $\pi_n(X)$. A base of the topology on $\pi_n(X)$ is given by the point-inverses of the maps $f_*: \pi_n(X) \rightarrow \pi_n(P)$ induced by all maps $f: X \rightarrow P$, where

P is a compact polyhedron. Clearly, group operations (when $n > 0$) as well as all induced maps are continuous in this topology. If X is the inverse limit of polyhedra P_i , by Lemma 2.1(a) every map f from X to a polyhedron Q extends to a map $P_{[k,\infty]} \rightarrow Q$ for some k , and so f_* factors through $(p_k^\infty)_*$. Consequently, a base of the topology on $\pi_n(X)$ can also be taken to consist of the point-inverses of all maps $(p_i^\infty)_*$ induced by the projections $p_i^\infty: X \rightarrow P_i$.

Clearly, the topological group (pointed space when $n = 0$) $\pi_n(X, x)$ is a shape invariant of the pair (X, x) . When X is Steenrod connected (i.e. $\pi_0(X) = pt$), it is also a shape invariant of X by Proposition 2.9. In addition, the underlying unpointed space of $\pi_0(X)$ is obviously a shape invariant of X .

Derived limit. Let $G = (\dots \xrightarrow{p_1} G_1 \xrightarrow{p_0} G_0)$ be an inverse sequence of groups. The group $\prod G_i$ acts on itself on the right by

$$(g_0, g_1, \dots) \cdot (x_0, x_1, \dots) = (x_0^{-1} g_0 p_0(x_1), x_1^{-1} g_1 p_1(x_2), \dots).$$

The stabilizer of 1 under this action can be identified with the inverse limit $\varprojlim G_i$. We define $\varprojlim^1 G_i$ to be the orbit space of this action, thought of as a pointed set. Note that the map $f: \prod G_i \rightarrow \prod G_i$ of pointed sets, defined by $f(g_0, g_1, \dots) = (g_0^{-1} p_0(g_1), g_1^{-1} p_1(g_2), \dots)$, is equivariant with respect to the right regular action of $\prod G_i$ on the domain and the above-defined action on the target. We have $\varprojlim G_i = \ker f$, and $\varprojlim^1 G_i$ vanishes iff f is surjective. When the G_i are abelian, f is a homomorphism, moreover, $\varprojlim^1 G_i$ can be identified with $\text{coker } f$ and thus becomes endowed with an abelian group structure.¹⁴

We endow $\varprojlim G_i$ with the inverse limit topology, which is induced from the product topology on the product $\prod G_i$ of the discrete groups; and $\varprojlim^1 G_i$ with the quotient topology, which is always anti-discrete since changing finitely many components of any $(g_1, g_2, \dots) \in \prod G_i$ does not change its orbit.

When the groups G_i are abelian, $G \mapsto \varprojlim^1 G_i$ is the derived functor of the inverse limit functor $G \mapsto \varprojlim G_i$ in the sense of homological algebra. For it can be shown that $\varprojlim^1 G_i$ vanishes if all bonding maps in G are epimorphisms, and therefore

$$\begin{array}{ccccccc} \dots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 \\ & & \left(\begin{smallmatrix} 1 \\ p_1 \\ p_0 p_1 \end{smallmatrix} \right) \downarrow & & \left(\begin{smallmatrix} 1 \\ p_0 \end{smallmatrix} \right) \downarrow & & 1 \downarrow \\ \dots & \longrightarrow & G_2 \times G_1 \times G_0 & \longrightarrow & G_1 \times G_0 & \longrightarrow & G_0 \\ & & \left(\begin{smallmatrix} p_1 & -1 & 0 \\ 0 & p_0 & -1 \end{smallmatrix} \right) \downarrow & & (p_0 \quad -1) \downarrow & & \downarrow \\ \dots & \longrightarrow & G_1 \times G_0 & \longrightarrow & G_0 & \longrightarrow & 0 \end{array}$$

is a \varprojlim -acyclic resolution of G (it is due to J. E. Roos). In the abelian case, Theorem 3.1(d) below is the usual long exact sequence for derived functors, and Theorem 4.1(iii) below (the homology version of Theorem 3.1(b)) is the universal coefficients formula for derived functors.

¹⁴If for each i and each $x_i \in G_i$ there exists a $j > i$ such that the image of every $y_j \in G_j$ in G_i commutes with x_i , then there is a monoid structure on $\varprojlim^1 G_i$; and if additionally each G_i is solvable, it extends to a group structure [Gü].

Theorem 3.1. (a) [Q2] If (X, A) is a pair of compacta with a basepoint $x \in A$ (or more generally $x: pt \rightsquigarrow A$), there is an exact sequence of pointed sets

$$\cdots \rightarrow \pi_2(X, A) \rightarrow \pi_1(A) \rightarrow \pi_1(X) \rightarrow \pi_1(X, A) \rightarrow \pi_0(A) \rightarrow \pi_0(X),$$

whose maps to the left of $\pi_1(X)$ (resp. $\pi_2(X)$) are homomorphisms of groups (resp. right $\mathbb{Z}\pi_1(A)$ -modules). In addition, $\pi_2(X) \rightarrow \pi_2(X, A)$ is also $\pi_1(A)$ -equivariant, $\partial: \pi_2(X, A) \rightarrow \pi_1(A)$ is a crossed module¹⁵, $\pi_1(X) \rightarrow \pi_1(X, A)$ is $\pi_1(X)$ -equivariant with respect to the right regular action on $\pi_1(X)$, and the non-trivial point-inverses of $\pi_1(X, A) \rightarrow \pi_0(A)$ coincide with the orbits of $\pi_1(X)$.

(b) [Q1], [Q2], [Gr], [EH; 5.2.1], [Wa] If (X, A) is the inverse limit of pairs (P_i, Q_i) of connected compact polyhedra, there is an exact sequence

$$1 \rightarrow \varprojlim^1 \pi_{n+1}(P_i, Q_i) \rightarrow \pi_n(X, A) \rightarrow \varprojlim \pi_n(P_i, Q_i) \rightarrow 1$$

of pointed sets when $n \geq 1$ or $A = Q_i = pt$ and $n = 0$, whose maps are group homomorphisms when $n \geq 2$ or $A = Q_i = pt$ and $n = 1$.

(c) The conclusion of (b) holds when (P_i, Q_i) are pairs of connected compacta.

(d) [BK; IX.2.3 (see also Ch. XI)], [MaS; p. 168] If $1 \rightarrow K_i \xrightarrow{j_i} G_i \xrightarrow{f_i} Q_i \rightarrow 1$ is an exact sequence of inverse sequences of groups (i.e. j_i and f_i commute with the bonding maps), there is an exact sequence of pointed sets

$$1 \rightarrow \varprojlim K_i \xrightarrow{\varprojlim j_i} \varprojlim G_i \xrightarrow{\varprojlim f_i} \varprojlim Q_i \xrightarrow{\delta} \varprojlim^1 K_i \xrightarrow{\varprojlim^1 j_i} \varprojlim^1 G_i \xrightarrow{\varprojlim^1 f_i} \varprojlim^1 Q_i \rightarrow 1,$$

where δ is $\varprojlim Q_i$ -equivariant with respect to the right regular action on $\varprojlim Q_i$ and the right action

$$[(k_0, k_1, \dots)] \cdot (g_0 K_0, g_1 K_1, \dots) = [(g_0^{-1} k_0 p_0(g_1), g_1^{-1} k_1 p_1(g_2), \dots)]$$

on $\varprojlim^1 K_i$, and the non-trivial point-inverses of $\varprojlim^1 j_i$ coincide with the orbits of $\varprojlim Q_i$.

(d') [Ir] If $1 \rightarrow H_i \xrightarrow{j_i} G_i \xrightarrow{f_i} G_i/H_i \rightarrow 1$ is an exact sequence of inverse sequences of pointed sets, where G_i and H_i are groups and j_i are homomorphisms, there is an exact sequence of pointed sets

$$1 \rightarrow \varprojlim H_i \xrightarrow{\varprojlim j_i} \varprojlim G_i \xrightarrow{\varprojlim f_i} \varprojlim (G_i/H_i) \xrightarrow{\delta} \varprojlim^1 H_i \xrightarrow{\varprojlim^1 j_i} \varprojlim^1 G_i,$$

where the non-trivial point-inverses of δ coincide with the orbits of the obvious action of $\varprojlim G_i$ on $\varprojlim (G_i/H_i)$.

The proof shows that all homomorphisms in Theorem 3.1 are natural and continuous.

¹⁵That is, $\partial(s \cdot g) = g^{-1}(\partial s)g$ for $g \in \pi_1(A)$, $s \in \pi_2(X, A)$ and $s^{-1}ts = t \cdot (\partial s)$ for $s, t \in \pi_2(X, A)$.

Proof. (a). This follows from the homotopy exact sequence for proper maps, which is verified directly. \square

(b). A direct verification shows that the following sequence of pointed sets of level-preserving pointed homotopy classes is exact to the left of the a arrow with $n = 1$ or with $Q_i = pt$ and $n = 0$:

$$\begin{aligned} \cdots \rightarrow [(D^{n+1}, S^n) \times \mathbb{N}, (P_{\mathbb{N}}, Q_{\mathbb{N}})]_{\ell}^* &\xrightarrow{b} [(D^n, S^{n-1}) \times [0, \infty), (P_{[0, \infty)}, Q_{[0, \infty)})]_{\ell}^* \\ &\xrightarrow{c} [(D^n, S^{n-1}) \times \mathbb{N}, (P_{\mathbb{N}}, Q_{\mathbb{N}})]_{\ell}^* \xrightarrow{a} \cdots \end{aligned}$$

“Pointed” means that $pt \times [0, \infty)$ always maps to $p_{[0, \infty)}$ and $pt \times \mathbb{N}$ always maps to $p_{\mathbb{N}}$. Here c is defined by restriction, and b using the obvious identification of $[(D^{n+1}, S^n) \times \mathbb{N}, (P_{\mathbb{N}}, Q_{\mathbb{N}})]_{\ell}^*$ with

$$[(D^n \times [0, \infty), S^{n-1} \times [0, \infty) \cup D^n \times \mathbb{N}), (P_{[0, \infty)}, Q_{[0, \infty)})]_{\ell}^*$$

for $n > 0$ or when $n = 0$ and $Q_i = pt$.

Finally, a may be identified with $f: \prod \pi_n(P_i, Q_i) \rightarrow \prod \pi_n(P_i, Q_i)$, given by $(g_1, g_2, \dots) \mapsto (p_{1*}(g_2) - g_1, p_{2*}(g_3) - g_2, \dots)$ in the abelian case (for $n \geq 3$ or $Q_i = pt$ and $n = 2$) and by $(g_1, g_2, \dots) \mapsto (g_1^{-1}p_{1*}(g_2), g_2^{-1}p_{2*}(g_3), \dots)$ in the group case (for $n \geq 2$ or $Q_i = pt$ and $n = 1$). In the pointed set case, a can be similarly defined if its target $[(D^1, S^0) \times \mathbb{N}, (P_{\mathbb{N}}, Q_{\mathbb{N}})]_{\ell}^*$ for $n = 1$ (respectively $[S^0 \times \mathbb{N}, P_{\mathbb{N}}]_{\ell}^*$ for $Q_i = pt$ and $n = 0$) is amended by moving the basepoint into $D^1 \setminus S^0$, contrary to the standard convention (respectively, is replaced with the basepoint free homotopy set $[S^0 \times \mathbb{N}, P_{\mathbb{N}}]_{\ell}$).

The maps in the long exact sequence are group homomorphisms to the left of $a: \prod \pi_n(P_i, Q_i) \rightarrow \prod \pi_n(P_i, Q_i)$ (non-inclusively!), where $n = 2$, or $Q_i = pt$ and $n = 1$. The assertion now follows since $\varprojlim \pi_n(P_i, Q_i) = \ker f$ and in the abelian case also $\varprojlim^1 \pi_{n+1}(P_i, Q_i) = \operatorname{coker} f$. In the non-abelian case, writing the rightmost homomorphism a as $a: G \rightarrow S$, we note that G acts on S by the formula in the definition of \varprojlim^1 , and a is equivariant with respect to this action on S and the right regular action on G . The orbits of this action are precisely the non-trivial point-inverses of the subsequent (to the right) map b . \square

(c). We suppress the basepoints. For the sake of clarity, the inverse sequence of compacta $\cdots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0$ will be denoted by $\cdots \xrightarrow{q_1} C_1 \xrightarrow{q_0} C_0$.

Let us represent each C_i as the inverse limit of $P^{(i)} = (\cdots \xrightarrow{p_1^{(i)}} P_1^{(i)} \xrightarrow{p_0^{(i)}} P_0^{(i)})$, where $P_j^{(i)}$ are compact polyhedra and $p_j^{(i)}$ are PL maps. The maps q^i induce Steenrod homotopy classes represented by proper maps $q_{[0, \infty)}^i: P_{[0, \infty)}^{(i+1)} \rightarrow P_{[0, \infty)}^{(i)}$. By Lemma 2.5(b₀), we may assume after passing to a subsequence of $P^{(1)}$ that $q_{[0, \infty)}^0$ is level-preserving. Continuing this procedure, we may assume that all $q_{[0, \infty)}^i$ are level-preserving. Let $T_J = (\cdots \xrightarrow{q_J^1} P_J^{(1)} \xrightarrow{q_J^0} P_J^{(i)})$, and let us write $T_{J \times K}$ for $(T_J)_K$. By the proof of (b), the exact sequence of (c) holds provided that $\pi_n(X)$ is replaced by the group G of fibered homotopy classes of maps $F: S^n \times [0, \infty) \times [0, \infty) \rightarrow T_{[0, \infty) \times [0, \infty)}$ that are fibered over $[0, \infty) \times [0, \infty)$.

Consider the diagonal “staircase” mapping telescope

$$T^{\Delta} = T_{[0, 1] \times 0} \cup T_{1 \times [0, 1]} \cup T_{[1, 2] \times 1} \cup T_{2 \times [1, 2]} \cup \cdots = Q_{[0, \infty)},$$

where $Q_{2i} = P_i^{(i)}$ and $Q_{2i+1} = P_{i+1}^{(i)}$. Clearly, X is homeomorphic to $\varprojlim Q_i$. On the other hand, restricting the fiberwise map F over T^Δ yields a level-preserving $F^\Delta: S^n \times [0, \infty) \rightarrow Q_{[0, \infty)}$, and conversely, every F^Δ extends to an F using the homotopies similar to Π_t . It follows that $G \simeq \pi_n(X)$. \square

(d). For variety, we include a geometric proof in the case where each G_i has a classifying space with compact 3-skeleton BG_i , and each Q_i has a classifying space with compact 4-skeleton BQ_i ; the general countable group case can be done similarly using the definitions at the end of §7. By considering the mapping cylinder, we may assume that $BG_i \subset BQ_i$, which inclusion induces the given surjection $G_i \rightarrow Q_i$ on π_1 . We have $\pi_2(BG_i) = 0 = \pi_2(BQ_i)$ and $\pi_3(BQ_i) = 0$, implying $\pi_3(BQ_i, BG_i) = 0$. Let $\cdots \rightarrow (BG_1, BQ_1) \rightarrow (BG_0, BQ_0)$ be an inverse sequence whose bonding maps induce the given homomorphisms $G_{i+1} \rightarrow G_i$ (and consequently also $Q_{i+1} \rightarrow Q_i$). Let $(X, A) = \varprojlim (BQ_i, BG_i)$. Then part (b) identifies the terms from the exact sequence of (a) with those from the statement of (d). The surjectivity of the rightmost map follows since in the notation of (b), if $\pi_0(P_i)$, $\pi_0(Q_i)$ and $\pi_1(P_i, Q_i)$ vanish, then $\pi_0(A) \rightarrow \pi_0(X)$ is surjective by the proof of (b). \square

(d'). Similar to (d); see also the proof of Lemma 3.7(a) below. \square

Example 3.2. As a sample application of Theorem 3.1(d), let us compute the group $\varprojlim^1(\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z})$. The given inverse sequence fits into a short exact sequence with $G = (\cdots \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z})$ and $Q = (\cdots \xrightarrow{\text{mod } p^2} \mathbb{Z}/p^2 \xrightarrow{\text{mod } p} \mathbb{Z}/p \rightarrow 1)$. Obviously $\varprojlim^1 G$ is trivial, so $\varprojlim^1(\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}) \simeq \varprojlim Q / \varprojlim G = \mathbb{Z}_p / \mathbb{Z}$.

The similarity between this calculation and the proof of Lemma 3.3 below will be analyzed and applied in the proof of Theorem 8.4.

An inverse sequence $\cdots \rightarrow G_1 \rightarrow G_0$ satisfies the *Mittag-Leffler condition*, if for each i there exists a $j > i$ such that for each $k > j$ the image of $G_k \rightarrow G_i$ equals that of $G_j \rightarrow G_i$. When the G_i are groups, it is an easy but enlightening exercise to verify that the Mittag-Leffler condition implies the vanishing of $\varprojlim^1 G_i$. The converse holds if each G_i is countable:

Lemma 3.3 [G], [Ge1], [DS1], [MM]. *Let G_i be an inverse sequence of countable groups. If $\varprojlim^1 G_i$ is countable (for example, trivial), then G_i satisfy the Mittag-Leffler condition.*

We include a proof since it will be needed later on several occasions. Our presentation is closest to [G], where Lemma 3.3 is stated under an additional hypothesis.

Proof. Suppose that the G_i are not Mittag-Leffler. Then there exists a k such that the images A_i of G_{k+i} in G_k do not stabilize. Since each $G_i \rightarrow A_i$ is onto, $\varprojlim^1 G_i \rightarrow \varprojlim^1 A_i$ is onto, so $\varprojlim^1 A_i$ is countable. If (g_0, g_1, \dots) and (h_0, h_1, \dots) are elements of $\prod A_i$ representing the same element of $\varprojlim^1 A_i$, each $g_i = x_i h_i x_{i+1}^{-1}$ for some $(x_0, x_1, \dots) \in \prod A_i$. Then each $g_0 \dots g_n = x_0 h_0 \dots h_n x_{n+1}^{-1}$. Hence $(g_0 A_1, g_0 g_1 A_2, \dots)$ and $(h_0 A_1, h_0 h_1 A_2, \dots)$ lie in the same orbit of the left action of A_0 on $A := \varprojlim A_0 / A_i$. As this argument is reversible, we obtain a bijection

between $\varprojlim^1 A_i$ and the orbit set $A_0 \backslash A$. However, A admits a bijection with the uncountable set $(A_0/A_1) \times (A_1/A_2) \times \dots$, hence $A_0 \backslash A$ must be uncountable. \square

An inverse sequence of pointed sets G_i is said to satisfy the *dual Mittag-Leffler condition*, if there exists a k such that for every $j > k$ there exists an $i > j$ for which the kernel from G_i to G_k equals that to G_j (cf. [Ge2]). If this condition holds, it is easy to see that $\varprojlim G_i$ injects into some G_k .

Lemma 3.4. *Let $\dots \rightarrow G_1 \rightarrow G_0$ be an inverse sequence of pointed sets. For (b) and (c), assume additionally that either it is an inverse sequence of groups or each G_i is finite.*

(a) *If the G_i satisfy the Mittag-Leffler condition and $\varprojlim G_i = 0$, then for each i there exists a $j > i$ such that $G_j \rightarrow G_i$ is trivial.*

(b) *If $\varprojlim G_i$ is discrete, it injects into some G_k .*

(c) *Let the G_i satisfy the Mittag-Leffler condition. If $\varprojlim G_i$ is discrete, then the G_i satisfy the dual Mittag-Leffler condition.*

The converse statements to (a), (b) and (c) are true, and obvious. The additional hypothesis for (b) and (c) can be dropped if $\varprojlim G_i$, understood as an inverse limit of discrete uniform spaces, is assumed to be discrete as a uniform space.

Part (b) will be needed a little later (in Theorem 3.10 and also in §6). Part (c) will be used in Theorem 3.12.

Proof. (a). Let L_i be the stable image of the G_j with $j > i$ in G_i . Then L_{i+1} maps onto L_i , and consequently $\varprojlim L_j$ maps onto L_i . The composition $\varprojlim L_j \rightarrow L_i \subset G_i$ factors through $\varprojlim G_i$, hence L_i is trivial. \square

(b). If g_1, g_2, \dots is a sequence of elements of $\varprojlim G_i$ such that each g_i maps trivially to G_i , then this sequence converges to the identity in the inverse limit topology. It remains to consider the case of finite sets. Suppose that $g_1, h_1, g_2, h_2, \dots$ is a sequence of elements of $\varprojlim G_i$ such that g_i and h_i map to the same element of G_i . Since $\varprojlim G_i$ is a compactum, there exists an infinite increasing sequence n_i such that g_{n_i} and h_{n_i} converge to some g and h respectively. Then $g = h$, and since $\varprojlim G_i$ is discrete, $g_{n_i} = g$ and $h = h_{n_i}$ for sufficiently large i . \square

(c). Let L_i be the stable image of the G_j with $j > i$ in G_i . Then $\varprojlim G_j$ maps onto L_i . By the hypothesis, there exists a k such that $\varprojlim G_j \rightarrow L_i$ is a bijection for all $i \geq k$. Given a $j > k$, let $i > j$ be such that G_i maps into L_j . Since $L_j \rightarrow L_k$ is a bijection, $\ker(G_i \rightarrow G_j) = \ker(G_i \rightarrow G_k)$. \square

Proposition 3.5. *Let X be a compactum, and fix some $m \geq 0$.*

(a) *X is the limit of an inverse sequence of m -connected polyhedra iff $\pi_n(X) = 0$ for all $n < m$ and $\pi_m(X)$ is anti-discrete.*

(b) *X is the limit of an inverse sequence of $(m+1)$ -connected maps between m -connected polyhedra iff $\pi_n(X) = 0$ for all $n \leq m$.*

We recall that a map $f: X \rightarrow Y$ is called *m -connected* if its mapping cylinder $MC(f)$ is relatively m -connected, i.e. $\pi_i(MC(f), X) = 0$ for $i \leq m$. By the homotopy exact sequence of a pair, a map $f: X \rightarrow Y$ between m -connected polyhedra is $(m+1)$ -connected iff $\pi_{m+1}(X) \rightarrow \pi_{m+1}(Y)$ is a surjection.

Compacta that satisfy any of the equivalent conditions of Proposition 3.5(a) are called UV_m or *approximatively m -connected* or *m -shape connected* in the literature. This notion is already found in [Ch; §5], along with the “only if” directions of both parts of Proposition 3.5. Part (a) is well-known (see e.g. [DS1; proof of 8.3.2]).

Proof. (a). The “only if” part is obvious. Conversely, suppose that X is the inverse limit of polyhedra P_i . By Theorem 3.1(b), $\lim_{\leftarrow} \pi_n(P_i) = 0 = \lim_{\leftarrow} {}^1\pi_n(P_i)$ for $n \leq m$. By Lemmas 3.3 and 3.4(a), for each i and each $n \leq m$ there exists a $j = j_n(i)$ such that $\pi_n(P_j) \rightarrow \pi_n(P_i)$ is trivial. If $j = j_0(j_1(\dots j_k(P_i)))$, then, by induction on k , the restriction of $P_j \rightarrow P_i$ to the k -skeleton $P_j^{(k)}$ of some fixed triangulation of P_j is null-homotopic. Set $i_0 = 0$ and $i_{k+1} = j_0(j_1(\dots j_m(i_k)))$ and let Q_k be the union of P_{i_k} with the cone $CP_{i_k}^{(m)}$. Then the bonding map $p_{i_k}^{i_{k+1}}: P_{i_{k+1}} \rightarrow P_{i_k}$ extends to a map $Q_{k+1} \rightarrow P_{i_k}$. Then X is the inverse limit of the maps $Q_{k+1} \rightarrow P_{i_k} \subset Q_k$. \square

(b). The “only if” part follows from Theorem 3.1(b). Conversely, suppose that X is the inverse limit of m -connected polyhedra P_i (using (a)) and that $G_i := \pi_{m+1}(P_i)$ is Mittag-Leffler (using Lemma 3.3). Thus for each i there exists a $j(i) > i$ such that for each $k > j$ the image of $p_i^k: G_k \rightarrow G_i$ equals that of p_i^j . Now G_j is finitely generated, using the Hurewicz theorem when $m > 0$. Let g_1, \dots, g_r be a set of generators. Then $p_i^j(g_l) = p_i^{j+1}(h_l)$ for some $h_1, \dots, h_r \in G_{j+1}$. Let $f_l: (S^{m+1}, pt) \rightarrow (P_j, p_j)$ be a spheroid representing $g'_l = g_l^{-1} p_i^{j+1}(h_l)$, and let Q_j be obtained by gluing up f_1, \dots, f_r by $(m+2)$ -cells. Then $P_j \rightarrow P_i$ extends to a map $Q_j \rightarrow P_i$, and the composition $P_{j+1} \rightarrow P_j \subset Q_j$ induces an epimorphism on π_{m+1} . Setting $i_0 = 0$ and $i_{n+1} = j(i_n)$, we have $X = \lim_{\leftarrow} Q_{i_n}$, where each Q_{i_n} is m -connected and each bonding map $Q_{i_{n+1}} \rightarrow Q_{i_n}$ is $(m+1)$ -connected. \square

Remark. By a well-known example depending on a deep result of J. F. Adams, Proposition 3.5 does not generalize to the case $m = \infty$, i.e. there exists an infinite-dimensional non-cell-like compactum X with $\pi_n(X) = 0$ for all n (see [EH; 5.5.10], [DS1; 10.3.1]).

The following “Whitehead theorem in Steenrod homotopy” answers a question of Koyama.

Theorem 3.6. *Let X, Y be connected compacta of dimensions $\leq m$ and $\leq m+1$ respectively, and let us consider a Steenrod homotopy class $f: (X, x) \rightsquigarrow (Y, y)$. If $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is a bijection for $n \leq m$ and a surjection for $n = m+1$, then f is a Steenrod homotopy equivalence.*

Just as with the classical Whitehead Theorem, the proof of Theorem 3.6 works to yield slightly more general assertions (a) and (b) below; addendum (c) follows from Theorem 3.6 itself along with the last assertion of addendum (b).

Addenda. *Let X and Y be connected compacta and $f: (X, x) \rightsquigarrow (Y, y)$ a Steenrod homotopy class.*

(a) *If $\dim X \leq m$, $\dim Y \leq m+1$, and $i: Y \hookrightarrow Z$ is an inclusion into a connected compactum Z such that the composition $\pi_n(X) \xrightarrow{f_*} \pi_n(Y) \xrightarrow{i_*} \pi_n(Z)$ is a bijection for $n \leq m$ and a surjection for $n = m+1$, then there exists a $g: Y \rightsquigarrow X$ such that $gf = [\text{id}_X]$ and $[i]fg = [i]$. In particular, $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is a split injection for all n .*

(b) If $\dim Y \leq m + 1$, and $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is a bijection for $n \leq m$ and a surjection for $n = m + 1$, then there exists a $g: Y \rightsquigarrow X$ such that $fg = [\text{id}_Y]$; and if $i: Z \hookrightarrow X$ is an inclusion of a connected compactum Z of dimension $\leq m$, then g can be chosen so that additionally $gf[i] = [i]$. In particular, $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is a split surjection for all n .

(c) If $\dim X \leq m$ and $\dim Y \leq m$, and $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is a bijection for all $n \leq m$, then f is a Steenrod homotopy equivalence.

A pro-group version of Theorem 3.6 appears in [EH; 5.5.6] with a sketch of proof; a different proof is given in [DS2] (see also [DS1] and [AM; §4]). Koyama used it to prove Addendum 3.6(c) under two additional hypotheses: (i) $f_*: \tilde{\pi}_n(X) \rightarrow \tilde{\pi}_n(Y)$ is a bijection for $n \leq m$; (ii) a technical condition which is slightly weaker than requiring Y to be an inverse limit of polyhedra with abelian fundamental groups [Koy]. He then asked whether these conditions (i) and (ii) are superfluous [Koy; Problem 2]. A hopelessly erroneous proof of Addendum 3.6(c) appears in [Li], where (without any comments) $\pi_1(P, Q)$ is thought to be a group for any polyhedral pair (P, Q) and then the vanishing of “ \varprojlim^1 ” of an inverse sequence of such “groups” is taken to imply the Mittag-Leffler condition. The correct part of the arguments found in [Li] amounts to a proof that condition (i) can be dropped from the hypothesis of Koyama’s theorem, but represents no progress with respect to the question of necessity of (ii).

Proof of Theorem 3.6. Let $f_{[0, \infty)}: P_{[0, \infty)} \rightarrow Q_{[0, \infty)}$ represent f . Without loss of generality all the P_i and Q_i are connected. By Lemma 2.5(b₀), after passing to a subsequence of P_i ’s we may assume that $f_{[0, \infty)}$ is level-preserving. Given any $J \subset [0, \infty)$, we write f_J for the restriction $P_J \rightarrow Q_J$ of f and M_J for the mapping cylinder of f_J . We may assume that $M_{[0, \infty)}$ is of dimension at most $m + 2$. From the hypothesis and the proof of Theorem 3.1(a) we get that $\pi_n(M_{[0, \infty)}, P_{[0, \infty)}) = 0$ for $1 \leq n \leq m + 1$. By the proof of Theorem 3.1(b) we have short exact sequences

$$0 \rightarrow \varprojlim^1 \pi_{n+1}(M_i, P_i) \rightarrow \pi_n(M_{[0, \infty)}, P_{[0, \infty)}) \rightarrow \varprojlim \pi_n(M_i, P_i) \rightarrow 0$$

for $n \geq 1$ and also that $\varprojlim^1 \pi_1(P_i) \rightarrow \varprojlim^1 \pi_1(M_i)$ is onto. Then $\pi_n(M_i, P_i)$ have trivial inverse limit for $1 \leq n \leq m + 1$ and trivial derived limit for $2 \leq n \leq m + 2$. By Lemma 3.3 and Lemma 3.7(b) below, the $\pi_n(M_i, P_i)$ are Mittag-Leffler for each n with $1 \leq n \leq m + 2$.

By Lemma 3.4(a), for each i there exists a j such that $\pi_n(M_j, P_j) \rightarrow \pi_n(M_i, P_i)$ is trivial for $1 \leq n \leq m + 1$. By passing to a subsequence, we may assume that $\pi_n(M_{i+1}, P_{i+1}) \rightarrow \pi_n(M_i, P_i)$ is trivial for $1 \leq n \leq m + 1$. We may also assume that the image of $\pi_{m+2}(M_{i+1}, P_{i+1})$ in $\pi_{m+2}(M_i, P_i)$ equals that of $\pi_{m+2}(M_j, P_j)$ for all $j > i$. Set $i_k = (2m + 3)k$ and let us fix some triangulation of each (M_{i_k}, P_{i_k}) . By an induction on $j = 0, 1, \dots, m + 1$, the bonding map $p: M_{i_k} \rightarrow M_{i_k-j}$ is homotopic rel P_{i_k} to a map φ_j sending the j -skeleton $M_{i_k}^{(j)}$ into P_{i_k-j} . Thus, writing $i'_k = i_k - m - 1$, the resulting map $\psi_k := \varphi_{m+1}$ sends M_{i_k} into $P_{i'_k}$, restricts to the bonding map $p: P_{i_k} \rightarrow P_{i'_k}$, and is homotopic to the bonding map $p: M_{i_k} \rightarrow M_{i'_k}$ by a homotopy Ψ_k .

Then $\mu: M_{i_{k+1}} \xrightarrow{p} M_{i_k} \xrightarrow{\psi_k} P_{i'_k}$ and $\nu: M_{i_{k+1}} \xrightarrow{\psi_{k+1}} P_{i'_{k+1}} \xrightarrow{p} P_{i'_k}$ are homotopic with values in $M_{i'_k}$ by a homotopy h_k . Writing $i''_k = i'_k - m - 1 = i_{k-1} + 1$, similarly to the above, the compositions of μ and ν with the bonding map $P_{i'_k} \rightarrow P_{i''_k}$ are

homotopic by a homotopy $h'_k: M_{i_{k+1}} \times I \rightarrow M_{i'_k}$ whose restriction to $M_{i_k}^{(m)} \times I$ has values in $P_{i'_k}$. For each $(m+1)$ -simplex Δ of $M_{i_{k+1}}$, we have $h'_k(\partial(\Delta \times I)) \subset P_{i'_k}$. Let $\alpha_\Delta: (\Delta \times I, \partial) \rightarrow (M_{i'_{k+1}}, P_{i'_{k+1}})$ be a relative spheroid whose composition with the bonding map $M_{i_{k+1}} \rightarrow M_{i_{k-1}}$ is homotopic in $(M_{i_{k-1}}, P_{i_{k-1}})$ to the composition of $h'_k|_{\Delta \times I}$ and the bonding map $M_{i'_k} \rightarrow M_{i_{k-1}}$. Amending ψ_{k+1} by all α_Δ 's, we obtain that the compositions of μ and the amended ν with the bonding map $P_{i'_k} \rightarrow P_{i_{k-1}}$ are homotopic (with values in $P_{i_{k-1}}$) by a homotopy h''_k . Thus the compositions

$g_{k+1}: M_{i_{k+1}} \xrightarrow{\psi_{k+1}} P_{i'_{k+1}} \xrightarrow{p} P_{i_{k-1}}$ are the integer slices of a level-preserving map $g_{[0,\infty)}: M_{[0,\infty)}^{is} \rightarrow P_{[0,\infty)}^i$, where $s(j) = j + 2$ and $M_j^i = M_{i_j}$, whose restriction to $P_{[0,\infty)}^{is}$ coincides with the shift $[p_{i_0}^{i_2}, p_{i_\infty}^{i_\infty}): P_{[0,\infty)}^{is} \rightarrow P_{[0,\infty)}^i$.

By construction¹⁶, $g_{[0,\infty)}$ is properly homotopic to $[p_{i_0}^{i_2}, p_{i_\infty}^{i_\infty}): M_{[0,\infty)}^{is} \rightarrow M_{[0,\infty)}^i$. Indeed, the amended ψ_{k+1} is homotopic to the original one by a homotopy Ψ'_{k+1} with values in $M_{i'_{k+1}}$. The homotopy h''_k is homotopic to the composition of h'_k and the bonding map $P_{i'_k} \rightarrow P_{i_{k-1}}$ by a 2-homotopy bounded by the compositions of Ψ'_k and Ψ'_{k+1} with the bonding maps. In turn, h'_k is relatively homotopic to the composition of h_k and the bonding map $P_{i'_k} \rightarrow P_{i'_k}$. Finally, h_k is homotopic to the bonding map $M_{i_{k+1}} \rightarrow M_{i'_k}$ by a 2-homotopy bounded by the compositions of Ψ_k and Ψ_{k+1} with the bonding maps. \square

Lemma 3.7. *Let G_i be an inverse sequence of groups and H_i an inverse sequence of their subgroups.*

(a) [Ir] *If the inverse sequence of the pointed sets G_i/H_i of right cosets satisfies the Mittag-Leffler condition, then $\varprojlim^1 H_i \rightarrow \varprojlim^1 G_i$ is onto.*

(b) *If all the G_i are countable, the converse holds.*

As the proof of Lemma 3.3 already uses both left and right actions, it seems unlikely that Lemma 3.7(b) can be similarly proved by a mere count of cardinalities. We use the Baire Category Theorem (see Corollary 7.11, where it is deduced from Bourbaki's "Mittag-Leffler Theorem").

Part (a) will be used a little later (in the proof of Theorem 3.15(b')).

Proof. (a). For variety, we indicate a geometric proof in the finitely presented case. Let $\cdots \rightarrow (P_2, Q_2) \rightarrow (P_1, Q_1)$ be an inverse sequence of pairs of compact connected polyhedra with $\pi_1(Q_i) = H_i$ and $\pi_1(P_i) = G_i$, the inclusion $H_i \subset G_i$ being induced by $Q_i \subset P_i$. Let $(X, A) = \varprojlim (P_i, Q_i)$ and write $S_i = \pi_1(P_i, Q_i)$. Assuming that $\text{im}[S_{i+1} \rightarrow S_i] = \text{im}[S_j \rightarrow S_i]$ for all i and all $j > i$, let us prove that $\pi_0(A) \rightarrow \pi_0(X)$ is surjective. Represent an element of $\pi_0(X, b)$, where $b \in A$, by a proper ray $b_{[0,1]} \ell_1 b_{[1,2]} \ell_2 \cdots: [0, \infty) \rightarrow P_{[0,\infty)}$, where $b_i = p_i^\infty(b)$ and the $\ell_i: (I, \partial I) \rightarrow (P_i, b_i)$ are loops. Then the path $b_{[0,1]} \ell_1 b_{[1,2]}$ is homotopic with values in $(P_{[0,2]}, Q_{[0,2]}; b_2)$ to a path $\ell'_1: (I, \partial I; \{0\}) \rightarrow (P_2, Q_2; b_2)$. Similarly, $\ell'_1 \ell_2 b_{[2,3]}$ is homotopic with values in $(P_{[1,3]}, Q_{[1,3]}; b_3)$ to a path $\ell'_2: (I, \partial I; \{0\}) \rightarrow (P_3, Q_3; b_3)$. Combining all such homotopies together yields a homotopy between the original proper ray and some proper ray with values in $Q_{[0,\infty)}$. \square

(b). Suppose that G_i/H_i is not Mittag-Leffler. Then there exists a k such that the images of G_{k+i}/H_{k+i} in G_k/H_k do not stabilize. Note that these are the same as the images of the compositions $G_{k+i} \rightarrow G_k \rightarrow G_k/H_k$. Let A_i be the image

¹⁶Or, alternatively, by (iii) \Rightarrow (i) in Proposition 2.8(b).

of G_{k+i} in G_k and let $B_i = A_i \cap H_k$. Then the images of A_i in A_0/B_0 do not stabilize; nor do the preimages $A_i B_0$ of these images. The anti-automorphism $g \mapsto g^{-1}$ sends them to $B_0 A_i$, and it follows that each pointed set $B_0 \backslash A_i / A_{i+1}$ of double cosets is nontrivial. Each $G_{k+i} \rightarrow A_i$ is onto, so $\varprojlim^1 G_{k+i} \rightarrow \varprojlim^1 A_i$ is onto. The composition $\varprojlim^1 H_{k+i} \rightarrow \varprojlim^1 G_{k+i} \rightarrow \varprojlim^1 A_i$ factors through $\varprojlim^1 B_i$, therefore $\varprojlim^1 B_i \rightarrow \varprojlim^1 A_i$ is surjective. Then the proof of Lemma 3.3 shows that $A := \varprojlim A_0 / A_i$ is the orbit of $B := \varprojlim B_0 / B_i$ under the left action of A_0 on A .

The inverse limit topology on A is induced by the ultrametric d defined by $d(x, y) = \frac{1}{n}$ whenever $p_n^\infty(x) = p_n^\infty(y) \in A_0/A_n$ but $p_{n+1}^\infty(x) \neq p_{n+1}^\infty(y) \in A_0/A_{n+1}$. Clearly A is complete in this metric (in fact it is the completion of $A_0/\bigcap A_i$). A ball of radius $\frac{1}{n}$ is a point-inverse of p_n^∞ . If such a ball $(p_n^\infty)^{-1}(gA_n)$ intersects B , then gA_n contains some $b \in B_0$. In this case let $B_0 a A_{n+1}$ be a nontrivial element of $B_0 \backslash A_n / A_{n+1}$. Then $ba A_{n+1}$ is contained in gA_n and is disjoint from B_0 . Hence $(p_{n+1}^\infty)^{-1}(ba A_{n+1})$ is contained in $(p_n^\infty)^{-1}(gA_n)$ and is disjoint from B . We proved that B is nowhere dense in A . Hence by the Baire Category Theorem, A cannot be the union of the countably many translates gB of B under the left action of A_0 on A by isometries. This is a contradiction. \square

Pontryagin considered the equivalence relation on inverse sequences of groups, generated by the operation of passing to a subsequence [P; Ch. III, §I]. Similarly to the proof of Proposition 2.6, (ii) \Leftrightarrow (iii), two inverse sequences G_i, H_i of abelian groups are equivalent in Pontryagin's sense iff they are related by a *pro-isomorphism*, which is a collection of maps $f_i: G_i \rightarrow H_i$ commuting with the bonding maps and such that there exist an increasing sequence $k: \mathbb{N} \rightarrow \mathbb{N}$ and homomorphisms $g_i: H_{k(i+1)} \rightarrow G_{k(i)}$ such that the compositions $g_i f_{k(i+1)}: G_{k(i+1)} \rightarrow G_{k(i)}$ and $f_{k(i)} g_i: H_{k(i+1)} \rightarrow H_{k(i)}$ equal the bonding maps.

Theorem 3.8. *A collection of homomorphisms $f_i: G_i \rightarrow H_i$ between inverse sequences of countable groups commuting with the bonding maps is a pro-isomorphism if and only if $\varprojlim f_i: \varprojlim G_i \rightarrow \varprojlim H_i$ and $\varprojlim^1 f_i: \varprojlim^1 G_i \rightarrow \varprojlim^1 H_i$ are bijections.*

This answers a question of Koyama [Koy; Problem 1], who proved the special case where each $f_i(G_i)$ is normal in H_i [Koy; Lemma 2]. The finitely generated abelian case was first announced by Keesling [Ke; Theorem 2.4].

Modulo Lemma 3.7(b), the proof of Theorem 3.8 is similar to that of [Koy; Lemma 2], but since one's reading of that proof is complicated by a misprint and several further references, for convenience we include the details.

Proof. The “only if” part follows since for any inverse sequence of groups Γ_i , the bonding maps induce bijections $\varprojlim \Gamma_{i+1} \rightarrow \varprojlim \Gamma_i$ and $\varprojlim^1 \Gamma_{i+1} \rightarrow \varprojlim^1 \Gamma_i$.

Conversely, each f_i yields short exact sequences $1 \rightarrow K_i \rightarrow G_i \rightarrow f_i(G_i) \rightarrow 1$ and $1 \rightarrow f_i(G_i) \rightarrow H_i \rightarrow L_i \rightarrow *$, where $K_i = \ker f_i$ and L_i is the pointed set $H_i/f(G_i)$ of right cosets. By the hypothesis and the six-term exact sequence 3.1(d,d'), $\varprojlim K_i = \varprojlim^1 K_i = 1$ and $\varprojlim L_i = *$. By Lemma 3.7(b), the L_i satisfy the Mittag-Leffler condition, and by Lemma 3.3 so do the K_i . Let $k(0) = 0$ and assume that $k(i)$ is defined. Then by Lemma 3.4(a), there exist $j > k(i)$ and $k(i+1) > j$ such that $L_{k(i+1)} \rightarrow L_j$ and $K_j \rightarrow K_{k(i)}$ are trivial. Given an $h \in H_{k(i+1)}$, its image in H_j equals $f_j(g_j)$ for some $g_j \in G_j$. Given any $g'_j \in G_j$ with $f_j(g'_j) = f_j(g_j)$, its image $g \in G_i$ equals the image of g_j . Hence $g_i: H_{k(i+1)} \rightarrow G_{k(i)}$, $h \mapsto g$, is well

defined. It is straightforward to verify that it is a homomorphism and that the g_i 's invert the h_i 's as required. \square

We now mention two known corollaries of Theorem 3.6; a third will appear in Theorem 3.15(e).

The following was proved in [DS2] and also follows from the π_∞ -criterion of Siebenmann [S1] as well as from [EH; 5.5.6].

Theorem 3.9. *Let X and Y be connected finite dimensional compacta. A Steenrod homotopy class $f: (X, x) \rightsquigarrow (Y, y)$ is a Steenrod homotopy equivalence iff it can be represented by a level-preserving map $f_{[0,\infty)}: (P_{[0,\infty)}, p_{[0,\infty)}) \rightarrow (Q_{[0,\infty)}, q_{[0,\infty)})$ that is an isomorphism in pro-homotopy.*

See Proposition 2.6 for the definition of an isomorphism in pro-homotopy. An interesting alternative proof of Theorem 3.9 is given in [Gü].

Proof. The “only if” part follows from Lemma 2.5(b). Conversely, the induced maps $(f_i)_*: \pi_n(P_i, p_i) \rightarrow \pi_n(Q_i, q_i)$ yield an isomorphism of pro-groups for each n . So by the “if” part of Theorem 3.8 and by Theorem 3.1(b), $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for all n . Hence by Theorem 3.6 f is a Steenrod homotopy equivalence. \square

If P is a non-compact polyhedron, then $\pi_n(P) := [(S^n, pt), (P, pt)]$ is isomorphic to $\hat{\pi}_n(P)$ by Proposition 2.3(a).

Theorem 3.10. (Edwards–Geoghegan) *Let X be an m -dimensional connected compactum. The following are equivalent:*

- (i) $\pi_n(X)$ is discrete for all $n \leq m$;
- (ii) X is pointed Steenrod homotopy dominated by a compact polyhedron;
- (iii) X admits a map into a polyhedron inducing isomorphisms on π_n for all $n \leq m$;
- (i') $\pi_n(X)$ is discrete for all n ;
- (ii') X is pointed Steenrod homotopy dominated by a compact m -polyhedron;
- (iii') X admits a map into an $(m+1)$ -polyhedron inducing isomorphisms on π_n for all n .

There is a number of different proofs of Theorem 3.10 in the literature. Though many of them are quite obscure from the geometric viewpoint: the original proof [EG1], [EG2] (see also a simplification in [D3]) involves non-separable polyhedra; Dydak [D2; 7.4] converts bonding maps to fibrations; Ferry [Fe2; Theorem 4 plus the first paragraph of its own proof] works with Hilbert cube manifolds. A closed geometric proof can be extracted from Kodama's papers [Ko2], [Ko3], which the present author learned of only when both proofs presented below had been written down. The second one may in fact be considered to be a simplified version of Kodama's argument.

Proof. (i) \Rightarrow (ii'). This will be proved by a combination of Addendum (a) to Theorem 3.6 with the trick in the proof of Proposition 3.5(b). An alternative proof will be given below based on Theorem 3.12.

Suppose that X is the limit of an inverse sequence of compact m -polyhedra P_i . Since $\pi_n(X)$ is Hausdorff for each $n \leq m$, by Theorem 3.1(b) and Lemma 3.3 the inverse sequence $\pi_n(P_i)$ is Mittag-Leffler for each $n \leq m+1$. By reindexing, we

may assume that for each i and each $n \leq m+1$, the image $G_i^{[n]}$ of $\pi_n(P_{i+1})$ in $\pi_n(P_i)$ equals the image of each $\pi_n(P_j)$ with $j > i$. Then each $G_{i+1}^{[n]}$ surjects onto $G_i^{[n]}$ and so does $\pi_n(X)$, for each $n \leq m+1$.

Since $\pi_n(X)$ is discrete for $n \leq m$, by Lemma 3.4(b) it injects into $\pi_n(P_k)$ for some k , which may be assumed to be the same for all $n \leq m$. Set $q = k + m + 1$. Since $G_q^{[1]}$ surjects onto $G_{q-1}^{[1]}$, each $x \in \pi_1(P_q)$ has the same image in $\pi_1(P_{q-1})$ with some $y_x \in G_q^{[1]}$. Attaching a 2-cell to P_q along some representative loop of xy_x^{-1} for each $x \in \pi_1(P_q)$, we obtain a (possibly noncompact) polyhedron $P_q^{[1]}$ such that the bonding map $P_q \rightarrow P_{q-1}$ factors through the inclusion $P_q \subset P_q^{[1]}$, and $\pi_1(X)$ maps onto $\pi_1(P_q^{[1]})$. If the cells are attached to $P_q \times [0, \infty)$ instead of to P_q , it is easy to keep the result locally compact. Next, since $\pi_2(P_{q-1})$ maps into $G_{q-2}^{[2]}$, so does $\pi_2(P_q^{[1]})$. Hence each $x \in \pi_2(P_q^{[1]})$ has the same image in $\pi_2(P_{q-2})$ as some element y_x of the image of $G_q^{[2]}$ in $\pi_1(P_q)$. Continuing the process, we will eventually factor the bonding map $P_q \rightarrow P_k$ through a sequence of inclusions $P_q \subset P_q^{[1]} \subset \dots \subset P_q^{[m+1]} =: Q$ such that $\pi_n(X)$ maps onto $\pi_n(Q)$ for each $n \leq m+1$.

Since $\pi_n(X)$ injects into $\pi_n(P_k)$ for all $n \leq m$, it also injects into $\pi_n(Q)$ for all $n \leq m$. Thus the triple $X \rightarrow P_q \rightarrow Q$ satisfies the hypothesis of Addendum (a) to Theorem 3.6, except that Q need not be compact. Nevertheless, the proof of Addendum 3.6(a) applies to show that P_q pointed Steenrod homotopy dominates X . \square

(ii) \Rightarrow (iii) and (ii') \Rightarrow (iii'). If K is a compact polyhedron and $d: (K, b) \rightsquigarrow (X, x)$ and $u: (X, x) \rightsquigarrow (K, b)$ are such that $du = [\text{id}_X]$, let us consider the doubly infinite mapping telescope $P := \text{Tel}(\dots \xrightarrow{q} K \xrightarrow{q} K \xrightarrow{q} \dots)$, where $q: (K, b) \rightarrow (K, b)$ is a PL map representing the Steenrod homotopy class ud by Proposition 2.3(a). From the Mather trick (see [FR]) and Proposition 2.3(a) we get a proper map $X \times \mathbb{R} \rightarrow P$ such that the composition $X \times \{0\} \subset X \times \mathbb{R} \rightarrow P$ induces an isomorphism on all π_n . \square

(i) \Rightarrow (i'), (ii) \Rightarrow (ii'), (iii) \Rightarrow (i), (iii') \Rightarrow (i'). Obvious. \square

Remark. As observed in [EG1], compacta satisfying the equivalent conditions of Theorem 3.10 need not have the shape of any compact polyhedron. Indeed, let P be a polyhedron that is homotopy dominated by a compact polyhedron K , but is not homotopy equivalent to any compact polyhedron (see [FR]). If $d: K \rightarrow P$ and $u: P \rightarrow K$ are such that $du \simeq \text{id}_P$, the limit X of the inverse sequence $\hat{K} := (\dots \xrightarrow{ud} K \xrightarrow{ud} K)$ is such that the projection $f: X \rightarrow P$ induces an isomorphism on Steenrod homotopy groups. Indeed, it is not hard to see that the level-preserving maps $d_{[0, \infty)}: \hat{K}_{[0, \infty)} \rightarrow P \times [0, \infty)$ and $u_{[0, \infty)}: P \times [0, \infty) \rightarrow \hat{K}_{[0, \infty)}$, defined respectively by d and u on the integer levels, and extended to all levels using the homotopies $d \simeq dud$ and $udu \simeq u$, are mutually inverse in semi-proper homotopy. Now if there exists a Steenrod homotopy equivalence $g: L \rightsquigarrow X$ for some compact polyhedron L , then $[f]g: L \rightsquigarrow P$ can be represented by a map $h: L \rightarrow P$ by Proposition 2.3(a). Since h induces isomorphisms of homotopy groups, it is a homotopy equivalence by the classical Whitehead theorem, which is a contradiction.

Remark. Ferry proved that, allowing the case $m = \infty$, for a connected compactum X , $\pi_n(X)$ is discrete for all $n < m$ iff X has the shape of a compactum that is LC_n

for all $n < m$ [Fe2]. The case $m = 1$ was first proved by Krasinkiewicz (see [DS1]); concerning the “if” part see §6. In particular, the following assertions can be added to the list of Theorem 3.10:

- (iv) X has the shape of an LC_m compactum;
- (iv') X has the shape of an LC_∞ compactum.

Lemma 3.11. (Dydak) *Let $A_i \rightarrow B_i \rightarrow C_i \rightarrow D_i$ be an exact sequence of inverse sequences of groups; or more generally of pointed sets where each A_i is a group acting on B_i so that the nonempty point-inverses of $B_i \rightarrow C_i$ are precisely the orbits of this action.*

If both the A_i and the C_i satisfy the Mittag-Leffler condition and the D_i satisfy the dual Mittag-Leffler condition, then the B_i satisfy the Mittag-Leffler condition.

Lemma 3.11 was proved in [D5], where it is stated in a weaker form (insufficient for its applications below). In §8 we shall need not only Lemma 3.11 but also its proof, so for convenience we reproduce it below, aiming at relative readability.

Proof. By reindexing, we may assume that $\text{im}(A_{i+1} \rightarrow A_i) = \text{im}(A_{i+2} \rightarrow A_i)$, $\text{im}(C_{i+1} \rightarrow C_i) = \text{im}(C_{i+2} \rightarrow C_i)$ and $\ker(D_i \rightarrow D_{i-1}) = \ker(D_i \rightarrow D_{i-2})$ for each i . It suffices to prove that $\text{im}(B_{n+2} \rightarrow B_n) = \text{im}(B_{n+3} \rightarrow B_n)$ for every n .

So let b_n be the image of some $b_{n+2} \in B_{n+2}$. Since $\text{im}(C_{n+2} \rightarrow C_{n+1}) = \text{im}(C_{n+4} \rightarrow C_{n+1})$, the image of b_{n+2} in C_{n+1} is the image of some $c_{n+4} \in C_{n+4}$. Then the image of c_{n+4} in D_{n+1} equals the image of b_{n+2} in D_{n+1} , which is trivial due to the exactness of the rows. Since $\ker(D_{n+4} \rightarrow D_{n+1}) = \ker(D_{n+4} \rightarrow D_{n+3})$, the image of c_{n+4} must be trivial already in D_{n+3} . Hence the image of c_{n+4} in C_{n+3} is the image of some $b_{n+3} \in B_{n+3}$. By construction, b_{n+2} and b_{n+3} map to the same element in C_{n+1} . Hence their images in B_{n+1} are related by the action by some $a_{n+1} \in A_{n+1}$. Since $\text{im}(A_{n+1} \rightarrow A_n) = \text{im}(A_{n+3} \rightarrow A_n)$, the image of a_{n+1} in A_n is the image of some $a_{n+3} \in A_{n+3}$. Finally, the result of the action of a_{n+3} on b_{n+3} maps onto b_n . \square

Remark. The above argument shows that for a fixed n , (i) if the images of A_i 's in A_n stabilize, and C_i are Mittag-Leffler and D_i are dual Mittag-Leffler, then the images of B_i 's in B_n stabilize; (ii) if the images of C_i 's in C_n stabilize, each $A_{i+1} \rightarrow A_i$ is an isomorphism, and D_i are dual Mittag-Leffler, then the images of B_i 's in B_n stabilize. This will be used in §8.

If P is a locally compact polyhedron, we say that P is *properly m -connected at infinity* if every compact subset Q of P is contained in a compact subset R of P such that every proper map $\mathbb{R}^n \rightarrow \text{Cl}(P \setminus R)$ with $n \leq m$ extends to a proper map $\mathbb{R}^n \times [0, \infty) \rightarrow \text{Cl}(P \setminus Q)$. Clearly, this property implies local n -connectedness of the one-point compactification of P . The converse implication does not hold (see Theorem 6.12a below).

Theorem 3.12. (Kodama) *Let X be the inverse limit of compact polyhedra P_i , and fix some $m \geq 0$. Then $\pi_n(X)$ is discrete for all $n < m$ if and only if $P_{[0, \infty)}$ is properly m -connected at infinity.*

Theorem 3.12 is essentially a restatement of some results of [Ko2], [Ko3]. Kodama's arguments are visibly simplified due to our use of Lemmas 3.11 and 3.13. The homological analogue of Theorem 3.12 was obtained by Dydak (see Theorem 6.12b).

Proof. We only consider the case $m > 0$. Theorem 3.1(a) furnishes the following exact sequence of inverse sequences:

$$\pi_n(X) \rightarrow \pi_n(P_{[k,\infty]}) \rightarrow \pi_n(P_{[k,\infty]}, X) \rightarrow \pi_{n-1}(X) \rightarrow \pi_{n-1}(P_{[k,\infty]}).$$

It is easy to see that $P_{[k,\infty]}$ Steenrod deformation retracts onto P_k , so their Steenrod homotopy groups (pointed sets when $n = 0$) can be identified.

If $\pi_n(X)$ is discrete for each $n < m$, then by Theorem 3.1(b) and Lemma 3.3, $\pi_n(P_k)$ satisfy the Mittag-Leffler condition for each $n \leq m$. By Lemma 3.4(c), $\pi_n(P_k)$ satisfy the dual Mittag-Leffler condition for each $n < m$. Hence by Lemma 3.11, $\pi_n(P_{[k,\infty]}, X)$ satisfy the Mittag-Leffler condition for each $n \leq m$. On the other hand, by Theorem 3.1(c), $\varprojlim \pi_n((P_{[k,\infty]}, X)) = \pi_n(X, X)$, which is trivial. Then by Lemma 3.4(a), for each i there exists a $j > i$ such that $\pi_n(P_{[j,\infty]}, X) \rightarrow \pi_n(P_{[i,\infty]}, X)$ is trivial for each $n \leq m$.

Conversely, suppose that for each i there exists a $j > i$ such that $\pi_n(P_{[j,\infty]}, X) \rightarrow \pi_n(P_{[i,\infty]}, X)$ is trivial. Then $\pi_{n-1}(X) \rightarrow \pi_{n-1}(P_j)$ is injective, from the exact sequence above. Hence $\pi_{n-1}(X)$ is discrete.

To complete the proof, we use Lemma 3.13 below. Clearly, a base-ray preserving proper map $\mathbb{R}^n \rightarrow P_{[k,\infty]}$ extends to a proper map $\mathbb{R}^n \times [0, \infty) \rightarrow P_{[k,\infty]}$ if and only if it is “null-homotopic” by a base-ray preserving proper homotopy. Moreover, every proper map $\mathbb{R}^n \rightarrow P_{[k,\infty]}$ is properly homotopic to a base-ray preserving one, as long as every base-ray preserving map $\mathbb{R}^1 \rightarrow P_{[k,\infty]}$ is “null-homotopic” by a base-ray preserving proper homotopy. \square

Lemma 3.13. *Let X be the inverse limit of compact polyhedra P_i . There is a natural pointed bijection between $\pi_n(P_{[k,\infty]}, X)$ and the set of base-ray preserving proper homotopy classes of base-ray preserving proper maps $\mathbb{R}^n \rightarrow P_{[k,\infty]}$.*

The inverse sequence of the pointed sets (groups for $n > 1$) $\pi_n(P_{[k,\infty]}, X)$ was considered by Kodama under the name “the movability pro-group” [Ko1]; more precisely, Kodama defined these by means of the statement of Lemma 3.13 and apparently did not realize that they coincide with the relative Steenrod homotopy sets (which were familiar to him).

Proof. By definition, $\pi_n(P_{[k,\infty]}, X)$ is the set of base-ray preserving proper homotopy classes of base-ray preserving proper maps $(\mathbb{R}^n \times [0, \infty), \mathbb{R}^n \times \{0\} \setminus B^n) \rightarrow (P_{[k,\infty]} \times [0, \infty), P_{[k,\infty]} \times \{0\})$, where $B^n \subset \mathbb{R}^n \times \{0\}$ is an n -ball (we omit base rays from the notation). Since $[0, \infty)$ deformation retracts onto $\{0\}$, we may assume without loss of generality that B^n does map into $P_{[k,\infty]} \times \{0\}$ under all maps and homotopies in question. This implies the assertion. \square

Theorem 3.12 immediately implies

Corollary 3.14. (Dydak [D4]) *If X and Y are compacta such that $\pi_i(X)$ and $\pi_i(Y)$ are discrete for all $i \leq n$ and $\pi_i(X \cap Y)$ are discrete for all $i \leq n - 1$ then $\pi_i(X \cup Y)$ are discrete for all $i \leq n$.*

Dydak’s proof uses homology, universal covers and nearly Steenrod connected compacta (concerning the latter, see Theorem 6.12). Under the stronger hypothesis that $\pi_i(X \cap Y)$ is discrete for all $i \leq n$, Corollary 3.14 was obtained by Kodama [Ko2], [Ko3]. The case $n = 0$ was obtained by Krasinkiewicz (see Theorem 8.3 below).

Alternative proof of Theorem 3.10, (i) \Rightarrow (ii'). In the terminology of §6, this will be the Steenrod-theoretic version of the Ferry Construction. Specifically, we will construct a Steenrod retraction $P_{[k,\infty]} \rightsquigarrow X$ for some k . It will follow that P_k Steenrod homotopy dominates X , for it is easy to see that $P_{[k,\infty]}$ Steenrod deformation retracts onto P_k .

Let us fix a triangulation of $P_{[0,\infty]}$, where the diameters of the simplices tend to zero as we approach X . Given some n and k_n , let $Q^n = (\cdots \rightarrow Q_{k_n+1}^n \rightarrow Q_{k_n}^n)$ denote the inverse sequence $\cdots \rightarrow P_{[k_n, k_n+2]}^{(n)} \rightarrow P_{[k_n, k_n+1]}^{(n)} \rightarrow P_{k_n}^{(n)}$, whose limit is the n -skeleton $P_{[k_n, \infty]}^{(n)}$. Then a Steenrod retraction $P_{[k_n, \infty]}^{(n)} \cup X \rightsquigarrow X$ is the proper homotopy class of a proper retraction $r^n: Q_{[0, \infty)}^n \cup P_{[0, \infty)} \rightarrow P_{[0, \infty)}$. Since $\pi_n(X)$ is discrete for all $n < 0$ (a vacuous condition), by Theorem 3.12, for each i there exists a $j = j(i) > i$ such that each vertex of $P_{[j, \infty)}$ is the endpoint of a proper ray $[0, \infty) \rightarrow P_{[i, \infty)}$. This yields an r^0 , with $k_0 = j(0)$. Assuming that r^n has been defined, we can define an r^{n+1} since by Theorem 3.12, for each i there exists a $j = j(i) > i$ such that every proper map $f: S^n \times [0, \infty) \rightarrow P_{[j, \infty)}$ with $f(S^n \times \{0\}) = \partial\Delta$ for some $(n+1)$ -simplex Δ of $P_{[j, \infty)}$ extends to a proper map $\bar{f}: B^{n+1} \times [0, \infty) \rightarrow P_{[i, \infty)}$ such that $\bar{f}(B^{n+1} \times \{0\}) = \Delta$. Finally, $[r^m]: P_{[k, \infty]} \rightsquigarrow X$ is the desired Steenrod retraction. \square

Steenrod fibrations. Let $P = (\cdots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0)$ and $Q = (\cdots \xrightarrow{q_1} Q_1 \xrightarrow{q_0} Q_0)$ be inverse sequences of compact polyhedra and PL maps and let $w: P \rightarrow Q$ be a *PL level map*, i.e. a sequence of PL maps $w_i: P_i \rightarrow Q_i$ such that $w_i p_i = q_i w_{i+1}$ for each i . We say that w satisfies the *homotopy lifting property* if for each i there exists a $j > i$ such that given a compact polyhedron K , a homotopy $f_t: K \rightarrow Q_j$ and a lift $\bar{f}_0: K \rightarrow P_j$ of \bar{f}_0 (i.e. $w_j \bar{f}_0 = f_0$), there exists a lift $\tilde{f}_t: K \rightarrow P_i$ of $q_i^j f_t$ (i.e. $w_i \tilde{f}_t = q_i^j f_t$) such that $\tilde{f}_0 = p_i^j \bar{f}_0$. Finally, a map $f: E \rightarrow B$ between compacta is called a *Steenrod fibration* if $E = \varprojlim P$, $B = \varprojlim Q$ and $f = \varprojlim w$ for some P , Q and w as above, with w satisfying the homotopy lifting property.

Clearly, the homotopy lifting property is satisfied, with $j(i) = i$, if

- (i) each f_i is a (Serre) fibration.

In particular, this is the case if

- (ii) each f_{i+1} is the composition of a fibration $\varphi_{i+1}: P_{i+1} \rightarrow R_{i+1}$ and the pullback $q_i^*(f_i): R_{i+1} \rightarrow Q_{i+1}$ of f_i .

In the latter situation (ii), f will be a fibration, as it is identified with the map ψ_0^∞ corresponding to the decomposition $E = \varprojlim (\cdots \xrightarrow{\psi_1} B_1 \xrightarrow{\psi_0} B_0 = B)$, where each ψ_i is a pullback of φ_i . In the more general situation (i), f does not have to be a fibration, as shown by the following examples.

The PL map $f: [0, 2] \rightarrow [0, 1]$, $x \mapsto \min(x, 1)$, is a Steenrod fibration (even with the f_i being fibrations) but not a fibration. Another such example is the projection of the topologist's sine curve (see Example 2.10) onto $[0, 1]$. On the other hand, if B is a compactum that contains no nondegenerate paths (for instance, the pseudo-arc), every map $f: E \rightarrow B$ is a fibration. As observed in [MR; Example 4], it does not have to be a Steenrod fibration, for instance, when $E = B \vee B$ (for any choice of a non-isolated basepoint in B) and f is the folding map.

Theorem 3.15. *Let $f: E \rightarrow B$ be a map between compacta.*

(a) Suppose that $E = \varprojlim R$ and $B = \varprojlim Q$, where $R = (\dots \xrightarrow{r_1} R_1 \xrightarrow{r_0} R_0)$ and $Q = (\dots \xrightarrow{q_1} Q_1 \xrightarrow{q_0} Q_0)$ are inverse sequences of compact polyhedra and PL maps. Then there exist an inverse sequence $P = (\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0)$ of compact polyhedra and PL maps with $\varprojlim P = E$ and PL level maps $u: P \rightarrow R$ and $w: P \rightarrow Q$ such that $\varprojlim u = \text{id}_E$ and $\varprojlim w = f$.

(b) [C2; 3.3] Suppose that f is a Steenrod fibration and $b \in B$ is such that $F := f^{-1}(b)$ is non-empty; pick some $\tilde{b} \in F$. Then there are natural actions of $\pi_1(E)$ on $\pi_n(F)$ for $n \geq 1$ and of $\pi_1(B)$ on $\pi_0(F)$ and a natural exact sequence of pointed sets

$$\dots \rightarrow \pi_1(F) \rightarrow \pi_1(E) \xrightarrow{f_*} \pi_1(B) \rightarrow \pi_0(F) \rightarrow \pi_0(E) \xrightarrow{f_*} \pi_0(B)$$

whose maps to the left of $\pi_1(B)$ (resp. $\pi_2(B)$) are homomorphisms of groups (resp. right $\mathbb{Z}\pi_1(E)$ -modules). In addition, $\pi_2(B) \rightarrow \pi_1(F)$ is also $\pi_1(E)$ -equivariant, $\pi_1(F) \rightarrow \pi_1(E)$ is a crossed module, $\pi_1(B) \rightarrow \pi_0(F)$ is $\pi_1(B)$ -equivariant with respect to the right regular action on $\pi_1(B)$, and the non-trivial point-inverses of $\pi_1(F) \rightarrow \pi_0(E)$ coincide with the orbits of $\pi_1(B)$.

(b') In the notation of (b), if B is connected, $f_*: \pi_0(E) \rightarrow \pi_0(B)$ is surjective.

(c) [C2; 3.4] If f is a Steenrod fibration and $b, b' \in B$ represent the same element of $\pi_0(B)$, then $f^{-1}(b)$ and $f^{-1}(b')$ have the same shape.

(d) If E is finite-dimensional and f is cell-like, that is, $f^{-1}(b)$ is cell-like for each $b \in B$, then f is a Steenrod fibration.

(e) [DS2] A cell-like map between connected finite-dimensional compacta is a Steenrod homotopy equivalence.

From part (a) it follows that the class of Steenrod fibrations coincides with the (a priori wider) class of shape fibrations, which were introduced in [MR].¹⁷

A weakened version of (c) was initially proved in [MR; Theorem 3]. Part (d) strengthens [MR; Theorem 5]. Weakened versions of (d) were initially obtained by Dydak (1975) and K. Morita (1975) (see [D2; 8.5]), improving earlier results by Bogatyj [Bog], Y. Kodama (1974) and Kuperberg [Ku1].

The assumption of finite dimensionality is necessary in (e) (see [DS1; 10.3.1]) and in (d) [MR; Example 6].

Proof. (a). Let $\Gamma \subset E \times B$ be the graph of f . Let P_i be a polyhedral neighborhood of the image of Γ in $R_i \times Q_i$ such that each P_{i+1} maps into P_i and $\varprojlim P_i = \Gamma$. Define $u_i: P_i \rightarrow R_i$ and $w_i: P_i \rightarrow Q_i$ to be the restrictions of the projections onto the factors of $R_i \times Q_i$. \square

(b). Let P, Q and w be as in the definition of a Steenrod fibration, with $j(i) = i + 1$ in the homotopy lifting property.

¹⁷Let us show more: If f is a shape fibration, and R, Q are as in (a), then there exist P, u and w as in (a) and such that w satisfies the homotopy lifting property. Indeed, by (a) we may assume that there is a PL level map $v: R \rightarrow Q$ with $\varprojlim v = f$. Then by [MR; Theorem 1], v satisfies the AHLP of [MR]. The required P, u and w are then furnished by the proof of [MR; Theorem 2].

Let us prove a homotopy lifting property for level-preserving representatives of Steenrod homotopy classes. Consider a polyhedron X and a level-preserving map $\varphi_{[0,\infty)}: X \times I \times [0,\infty) \rightarrow Q_{[0,\infty)}$ along with a lift $\bar{\varphi}_{[0,\infty)}: X \times [0,\infty) \rightarrow P_{[0,\infty)}$ of its restriction to $X \times \{0\} \times [0,\infty)$. By the hypothesis, every its shifted level $q_{i-1}\varphi_i$ lifts to a map $\tilde{\varphi}_{i-1}: X \times I \rightarrow P_i$ restricting to $p_{i-1}\bar{\varphi}_i$ on $X \times \{0\}$. Since $q_{[i-2,i-1]}^{[i,i+1]}: Q_{[i,i+1]} \rightarrow Q_{[i-2,i-1]}$ factors through $q_{i-1} \times \text{id}_I: Q_i \times I \rightarrow Q_{i-1} \times I$, we also obtain a lift $\tilde{\varphi}_{[i-2,i-1]}$ of $q_{[i-2,i-1]}^{[i,i+1]}\varphi_{[i,i+1]}$, restricting to $p_{[i-2,i-1]}^{[i,i+1]}\bar{\varphi}_{[i,i+1]}$ on $X \times \{0\} \times [i, i+1]$. Two consecutive instances of this construction yield two possibly different lifts $\tilde{\varphi}'_{i-2}$ and $\tilde{\varphi}''_{i-2}$ of $q_{i-2}^i\varphi_i$ restricting to $p_{i-2}^i\bar{\varphi}_i$ on $X \times \{0\}$. By the hypothesis, $p_{i-3}\tilde{\varphi}'_{i-2}$ is homotopic to $p_{i-3}\tilde{\varphi}''_{i-2}$ through lifts of $q_{i-3}^i\varphi_i$ that restrict to $p_{i-3}^i\bar{\varphi}_i$ on $X \times \{0\}$. It follows that $p_{[i-3,i-2]}\tilde{\varphi}_{[i-2,i-1]}$ is homotopic with support over $[i-3, i-2.5]$ to a new lift $\tilde{\varphi}'_{[i-3,i-2]}$ of $q_{[i-3,i-2]}^{[i,i+1]}\varphi_{[i,i+1]}$ that now agrees with its neighbor $\tilde{\varphi}'_{[i-4,i-3]}$. Thus we obtain a lift $\tilde{\varphi}_{[0,\infty)}: X \times [0,\infty) \rightarrow P_{[0,\infty)}$ of $q_{[-3,\infty-3]}^{[0,\infty)}\varphi_{[0,\infty)}$ restricting to $p_{[-3,\infty-3]}^{[0,\infty)}\bar{\varphi}_{[0,\infty)}$ on $X \times \{0\} \times [0,\infty)$.

The thus established property easily implies that the inclusion induced map $\pi_n(CF, F; \tilde{b}) \rightarrow \pi_n(MC(f), E; \tilde{b})$ is a bijection for $n \geq 1$.¹⁸ Since CF is cell-like, being an inverse limit of cones, $\partial: \pi_n(CF, F; \tilde{b}) \rightarrow \pi_{n-1}(F; \tilde{b})$ is a bijection by Theorem 3.1(a). Hence the assertion follows from the homotopy exact sequence of $(MC(f), E)$, given by Theorem 3.1(a). \square

(b'). We may assume that E is connected; else we may replace it with the connected component containing \tilde{b} and verify that the restriction of f is a Steenrod fibration. Let P, Q and w be as in the definition of a Steenrod fibration, with $j(i) = i+1$ in the homotopy lifting property. Then by Theorem 3.1(b), $\pi_0(E) \rightarrow \pi_0(B)$ can be identified with $\varprojlim (w_i)_*: \varprojlim \pi_1(P_i) \rightarrow \varprojlim \pi_1(Q_i)$. Without loss of generality, all the P_i are connected, so the cokernel of $(w_i)_*: \pi_1(P_i) \rightarrow \pi_1(Q_i)$ can be identified with $S_i := \pi_0(MC(w_i), P_i)$. By Lemma 3.7(a), it suffices to show that the inverse sequence $\cdots \rightarrow S_1 \rightarrow S_0$ satisfies the Mittag-Leffler condition. Indeed, let $F_i = w_i^{-1}(b_i)$, where $b_i = p_i^\infty(b)$. By the homotopy lifting property, each bonding map $S_{i+1} \rightarrow S_i$ factors through $T_i := \pi_1(MC(w_i|_{F_i}), F_i)$. Each T_i can be identified with $\pi_0(F_i)$ and so is finite. Hence S_i satisfy the Mittag-Leffler condition. \square

(c). Let P, Q and w be as in the definition of a Steenrod fibration, with $j(i) = i+1$ in the homotopy lifting property. Let $R_i = w_i^{-1}(b_i)$ and $R'_i = w_i^{-1}(b'_i)$, where $b_i = q_i^\infty(b)$ and $b'_i = q_i^\infty(b')$, and let $v: R_{[0,\infty)} \rightarrow [0,\infty)$ be the level-preserving projection. Pick a level-preserving Steenrod path $\ell_t: [0,\infty) \rightarrow Q_{[0,\infty)}$ connecting $b_{[0,\infty)}$ with $b'_{[0,\infty)}$. Then by the proof of (b), the proper homotopy $q_{[-3,\infty-3]}^{[0,\infty)}v\ell_t: R_{[0,\infty)} \rightarrow Q_{[0,\infty)}$ lifts to a homotopy $h_t: R_{[0,\infty)} \rightarrow P_{[0,\infty)}$ such that h_0 is the inclusion composed with $p_{[-3,\infty-3]}^{[0,\infty)}$. In particular, we obtain a proper map $h_1: R_{[0,\infty)} \rightarrow R'_{[0,\infty)}$; similarly, one constructs $h'_1: R'_{[0,\infty)} \rightarrow R_{[0,\infty)}$. Since the folding map $[-1, 1] \rightarrow [0, 1]$, $x \mapsto |x|$, is null-homotopic keeping the endpoints fixed, $h_1h'_1$ and h'_1h_1 are properly homotopic to the identity. \square

¹⁸Another possible approach is to observe that $f_*: \pi_n(E, F; \tilde{b}) \rightarrow \pi_n(B; b)$ is a bijection. This leads to weaker conclusions: the exact sequence becomes shorter by one term; the $\pi_1(E)$ -actions reduce, via the restriction of scalars, to $\pi_1(F)$ -actions; and the $\pi_1(B)$ -action reduces to a $\pi_1(E)$ -action.

(d). Let P , Q and w be as in (a), with $\dim P_i \leq m$ for each i . Let us triangulate each Q_i so that each q_i is simplicial with respect to the triangulation of Q_{i+1} and some subdivision of the second barycentric subdivision of the triangulation of Q_i . Pick a point $v \in B$, and for each i let N_i^v be the simplicial neighborhood of $q_i^\infty(v)$ in Q_i . Then each $q_i(N_{i+1}^v) \subset N_i^v$ and $\varprojlim N_i^v = \{v\}$. Moreover, the union of all N_{i+1}^u 's that intersect N_{i+1}^v is mapped into N_i^v due to our choice of the triangulations. Let $M_i^v = w_i^{-1}(N_i^v)$. Using that $f^{-1}(v) = \varprojlim M_i^v$ is cell-like and that $\dim P_i \leq m$, by the proof of Proposition 3.5(a) we may assume that each $p_i|_{M_{i+1}^v}: M_{i+1}^v \rightarrow M_i^v$ is null-homotopic.

Let $Q'_i = \bigcup_{v \in B} N_i^v$, in other words the simplicial neighborhood of $q_i^\infty(B)$, and let $P'_i = \bigcup_{v \in B} N_i^v \times M_i^v \subset Q'_i \times P_i$ (note that the union is finite). Let $p'_i = (q_i \times p_i)|_{P'_i}$ and $q'_i = q_i|_{Q'_i}$, and let $w'_i: P'_i \rightarrow Q'_i$ be the restriction of the projection. Given a linear map $\varphi: \Delta^{n+1} \rightarrow Q'_{i+2}$ and a partial lift $\bar{\varphi}_\partial: \partial\Delta^{n+1} \rightarrow P'_{i+2}$, we have $\varphi(\Delta^{n+1}) \subset N_{i+2}^v$ for some $v \in X$ and therefore $\bar{\varphi}_\partial(\partial\Delta^{n+1})$ lies in the union of all products $N_{i+2}^u \times M_{i+2}^u$ where N_{i+2}^u intersects N_{i+2}^v . Then $p'_{i+1}\bar{\varphi}_\partial(\partial\Delta^{n+1}) \subset N_{i+1}^v \times M_{i+1}^v$. Let $\tilde{\varphi}_\partial: \partial\Delta^{n+1} \rightarrow M_{i+1}^v$ be the composition of $p'_{i+1}\bar{\varphi}_\partial$ and the projection. Then the shift $p_i\tilde{\varphi}_\partial: \partial\Delta^{n+1} \rightarrow M_i^v$ extends to a map $\tilde{\varphi}: \Delta^{n+1} \rightarrow M_i^v$. Let $\bar{\varphi} := (q_i^{i+2}\varphi) \times \tilde{\varphi}: \Delta^{n+1} \rightarrow N_i^v \times M_i^v$. Thus $\bar{\varphi}: \Delta^{n+1} \rightarrow P'_i$ is a lift of $q'_i q'_{i+1}\varphi$ extending $p'_i p'_{i+1}\bar{\varphi}_\partial$. In particular, $w': P' \rightarrow Q'$ satisfies the homotopy lifting property. \square

(e). This follows from parts (b) and (d) combined with Theorem 3.6. \square

4. HOMOLOGY AND COHOMOLOGY

Steenrod homology. Following [St; §IV], the group $H_n(X, A)$ is defined to be the locally finite homology group $H_{n+1}^{\text{lf}}(P_{[0,\infty)}, P_0 \cup Q_{[0,\infty)})$, where (X, A) is the inverse limit of the compact polyhedral pairs (P_i, Q_i) . We recall that the locally finite homology $H_i^{\text{lf}}(P_{[0,\infty)}, P_0 \cup Q_{[0,\infty)})$ can be defined as the group of oriented proper pseudo-bordism classes of proper maps from oriented i -pseudo-manifolds $(M, \partial M)$ into $(P_{[0,\infty)}, P_0 \cup Q_{[0,\infty)})$. Since locally finite homology is an invariant of proper homotopy type, $H_n(X, A)$ is a shape invariant of the pair (X, A) by Corollary 2.2. (This in particular entails the topological invariance of Steenrod homology.) The induced maps f_* , where f is a map or a Steenrod homotopy class, and the boundary maps δ_* (which are also Steenrod homotopy invariants) are defined in the obvious ways.

Theorem 4.1. (Milnor [Mi]) *Steenrod homology satisfies all Eilenberg–Steenrod axioms, as well as the following:*

(i) *Map Excision Axiom:* $f_*: H_n(X, A) \rightarrow H_n(Y, B)$ is an isomorphism for any map $f: (X, A) \rightarrow (Y, B)$ that restricts to a homeomorphism $X \setminus A \rightarrow Y \setminus B$.

(ii) *Cluster Axiom:* $H_n((\bigsqcup X_i \setminus pt)^+, pt) \simeq \prod H_n(X_i, pt)$ naturally for any compacta X_1, X_2, \dots , where Y^+ denotes the one-point compactification of Y .

(iii) *Milnor's Exact Sequence:* if $(X, A) = \varprojlim (\dots \xrightarrow{q^1} (Y_1, Z_1) \xrightarrow{q^0} (X_0, Z_0))$, where (Y_i, Z_i) are pairs of compacta, there is a natural short exact sequence of groups

$$0 \rightarrow \varprojlim H_{n+1}(Y_i, Z_i) \rightarrow H_n(X, A) \rightarrow \varprojlim H_n(Y_i, Z_i) \rightarrow 0.$$

(iv) Steenrod homology is the only ordinary homology theory (in the sense of the Eilenberg–Steenrod axioms) on pairs of compacta that satisfies (ii) for compact polyhedra X_i and (i).

Note that (ii) is a special case of (iii), using that the cluster $(\bigsqcup X_i \setminus pt)^+$ is homeomorphic to $\varprojlim (\cdots \rightarrow X_1 \vee X_2 \vee X_3 \rightarrow X_1 \vee X_2 \rightarrow X_1)$.

A version of (iii) is found already in [St; Theorem 8].

Pontryagin cohomology. The group $H^n(X, A)$ is defined to be the compactly supported cohomology group $H_c^{n+1}(P_{[0,\infty)}, P_0 \cup Q_{[0,\infty)})$, where (X, A) is the inverse limit of compact polyhedral pairs (P_i, Q_i) . Remarks similar to those about $H_n(X, A)$ apply, and $H_n(X, A; G)$ can be similarly defined. In addition, we have

$$H_c^{n+1}(P_{[0,\infty)}, P_0) \simeq \varinjlim H^{n+1}(P_{[0,\infty)}, P_0 \cup P_{[k,\infty)}) \simeq \varinjlim H^n(P_{[k,\infty)}) \simeq \varinjlim H^n(P_k)$$

where the second isomorphism uses that $H^{n+1}(P_{[0,\infty)}, P_0) = 0$ since $P_{[0,\infty)}$ deformation retracts onto P_0 ; and the third that $P_{[k,\infty)}$ deformation retracts onto P_k . The composite isomorphism

$$H^n(X) \simeq \varinjlim H^n(P_k)$$

constitutes the original definition of $H^n(X)$, which for finite-dimensional compacta is due to Pontryagin [P; Ch. III, §II]. As a corollary of this isomorphism and Proposition 2.3(b), we obtain

Proposition 4.2. *If G is a countable abelian group, there is a natural isomorphism $H^n(X; G) \simeq [X, K(G, n)]$.*

The cohomological analog of Theorem 4.1 holds, with the direct product in (ii) replaced by the direct sum, and with the exact sequence of (iii) replaced by a natural isomorphism $H^n(X, A) \simeq \varinjlim H^n(Y_i, Z_i)$ [Mi].

Theorem 4.3. *If $X \subset \mathbb{R}^m$ is compact, there are natural (with respect to inclusion) isomorphisms $H^n(X) \simeq \tilde{H}_{m-n-1}(\mathbb{R}^m \setminus X)$ and $H_n(X) \simeq \tilde{H}^{m-n-1}(\mathbb{R}^m \setminus X)$.*

The second isomorphism is due to Steenrod [St]. The first isomorphism traces back to the original duality theorem of Alexander (1922), which was stated for mod2 Betti numbers and for polyhedra X (possibly wildly embedded). These restrictions were gradually removed by Alexandroff, Frankl, Lefschetz and eventually by Pontryagin (see [P] and references there).

It is well-known that upon adoption of the map excision axiom these isomorphisms become obvious, modulo the Poincaré duality on the non-compact manifold $S^m \setminus X$ (see e.g. [Fe3]). Since the verification of the map excision axiom takes more than a few lines, it seems worthwhile to be also aware of a direct version of this argument:

Proof. Let $\cdots \subset U_2 \subset U_1 \subset U_0$ be a sequence of polyhedral neighborhoods of X with $\bigcap U_i = X$. Then the definition of $H_n(X)$ and an appropriate kind of the Poincaré duality yield

$$H_n(X) \simeq H_{n+1}^{\text{lf}}(U_{[0,\infty)}, U_0) \simeq H^{m-n}(U_{[0,\infty)}, \text{Fr } U_{[0,\infty)}).$$

Using excision, the exact sequence of a pair, and the vanishing of $\tilde{H}^i(\mathbb{R}^m \times [0, \infty))$, the latter group is isomorphic to

$$\tilde{H}^{m-n-1}(\mathbb{R}^m \times [0, \infty) \setminus U_{[0, \infty)}) \simeq \tilde{H}^{m-n-1}(\mathbb{R}^m \setminus X).$$

The latter isomorphism holds since the projection $\mathbb{R}^m \times [0, \infty) \setminus U_{[0, \infty)} \rightarrow \mathbb{R}^m \setminus X$ is a homotopy equivalence. Similarly,

$$\begin{aligned} H^n(X) &\simeq H_c^{n+1}(U_{[0, \infty)}, U_0) \simeq H_{m-n}(U_{[0, \infty)}, \text{Fr } U_{[0, \infty)}) \\ &\simeq \tilde{H}_{m-n-1}(\mathbb{R}^m \times [0, \infty) \setminus U_{[0, \infty)}) \simeq \tilde{H}_{m-n-1}(\mathbb{R}^m \setminus X). \quad \square \end{aligned}$$

Remark. The above argument yields an elementary proof of the Jordan curve theorem, using only basic facts about (co)homology of polyhedra; the Poincaré duality; and Lemma 2.1. (The latter is needed since $H^1(S^1) = H_c^2(P_{[0, \infty)}, P_0)$ is computed twice: with $P_i = U_i$ and with $P_i = S^1$.)

By analogy with the direct limit interpretation of Pontryagin cohomology, it is convenient to reserve a special notation for the inverse limits that appear in Milnor's short exact sequences:

Čech homotopy and homology. If X is the inverse limit of polyhedra P_i , consider the topological group (pointed space for $n = 0$) $\check{\pi}_n(X, x) := \varprojlim \pi_n(P_i, p_i)$ and the topological group $\check{H}_n(X) := \varprojlim H_n(P_i)$. By Proposition 2.6, $\check{\pi}_n$ (resp. \check{H}_n) and indeed the entire short exact sequence 3.1(b) (resp. 4.1(iii)) is a shape invariant of (X, x) (resp. X). The canonical epimorphisms $\pi_n(X) \rightarrow \check{\pi}_n(X)$ and $H_n(X) \rightarrow \check{H}_n(X)$ will be denoted by $\check{\tau}$.

Here is the (absolute) “Hurewicz theorem in Steenrod homotopy”.

Theorem 4.4. *Let $m > 1$ and let X be a compactum with $\pi_n(X) = 0$ for all $n < m$. Then the Hurewicz homomorphism $\pi_n(X) \rightarrow H_n(X)$ is an isomorphism for $n \leq m$ and an epimorphism for $n = m + 1$.*

The first assertion of Theorem 4.4 was conjectured in [Ch; p. 300] and proved 40 years later by Lisitsa [Li]. His proof of the second assertion, however, contains an error: in the case $m = 2$ it depends on the claim that if X is the inverse limit of polyhedra P_i , then already $\varprojlim^1 \pi_{m+2}(P_i) \rightarrow \varprojlim^1 H_{m+2}(P_i)$ is allegedly an epimorphism. But this claim is false for $m = 2$. Indeed, let P_i be the cone of the composition $S^3 \xrightarrow{p^i} S^3 \xrightarrow{h} S^2$ of a degree p^i map and the Hopf map. The identity on S^2 extends to a map $f_i: P_{i+1} \rightarrow P_i$; let $X = \varprojlim (\cdots \rightarrow P_2 \rightarrow P_1)$. Clearly each $(f_i)_*: H_4(P_{i+1}) \rightarrow H_4(P_i)$ is the inclusion $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ onto the index p subgroup, so $\varprojlim^1 H_4(P_i) \neq 0$. Since P_i is simply-connected, every map $S^4 \rightarrow P_i$ is homotopic to one where the cone vertex has precisely one point-inverse; it follows that $\pi_4(hp^i)$ surjects onto $\pi_4(P_i)$. From the exact sequence

$$\pi_4(S^2) \rightarrow \pi_4(hp^i) \rightarrow \pi_3(S^3) \xrightarrow{hp^i} \pi_3(S^2),$$

$\pi_4(hp^i)$ is a quotient of $\pi_4(S^2) \simeq \mathbb{Z}/2$. So each $\pi_4(P_i)$ is finite and $\varprojlim^1 \pi_4(P_i) = 0$. Thus $\varprojlim^1 \pi_4(P_i) \rightarrow \varprojlim^1 H_4(P_i)$ cannot be surjective.

Remark. The problem can be fixed in this example by using the surjectivity of the connecting homomorphism $\lim_{\leftarrow} \pi_3(P_i) \rightarrow \lim_{\leftarrow} {}^1H_4(P_i)$ for the short exact sequence of inverse sequences

$$0 \rightarrow H_4(P_i) \rightarrow \Gamma_3(P_i) \rightarrow \pi_3(P_i) \rightarrow 0$$

obtained from Lemma 4.5 below. Indeed, each $\Gamma_3(P_{i+1}) \rightarrow \Gamma_3(P_i)$ is an isomorphism (between copies of \mathbb{Z}). Hence $\lim_{\leftarrow} {}^1\Gamma_3(P_i) = 0$ and so the connecting homomorphism must be surjective.

Lemma 4.5. (Whitehead [Wh]) *Let K be a simply-connected polyhedron with a fixed triangulation, and let $\Gamma_n(K)$ be the image of $\pi_n(K^{(n-1)}) \rightarrow \pi_n(K^{(n)})$. The Hurewicz homomorphisms fit into a long exact sequence*

$$\cdots \rightarrow \Gamma_n(K) \rightarrow \pi_n(K) \rightarrow H_n(K) \rightarrow \Gamma_{n-1}(K) \rightarrow \cdots$$

The use of geometric language enables us to simplify Whitehead's argument.

Proof. The assertion of the lemma becomes obvious if Γ_n is replaced with the group Γ'_n of ∂ -spherical singular oriented pseudo-bordism classes of ∂ -spherical singular oriented $(n+1)$ -pseudo-manifolds in K . (An n -pseudo-manifold M with boundary is ∂ -spherical if ∂M is PL homeomorphic to S^{n-1} . A pseudo-bordism W between such M_0, M_1 is ∂ -spherical if the closure of $\partial W \setminus (\partial M_0 \cup \partial M_1)$ is PL homeomorphic to $S^{n-1} \times I$. The group operation on pseudo-bordism classes is given by boundary connected sum.) Since K is simply-connected, Γ'_n is isomorphic to the group Γ''_n of simply-connected ∂ -spherical singular oriented pseudo-bordism classes of simply-connected ∂ -spherical singular oriented $(n+1)$ -pseudo-manifolds in K .

If M is a ∂ -spherical oriented $(n+1)$ -pseudo-manifold, it collapses onto a union L of $M^{(n-1)}$ with some n -simplices of M . If additionally M is simply-connected, the Hurewicz homomorphism $\pi_n(L, L^{(n-1)}) \rightarrow H_n(L, L^{(n-1)})$ is an isomorphism. Next, $[\partial M] \in H_n(M)$ is trivial, in particular it maps trivially to $H_n(L, L^{(n-1)})$. It follows that the inclusion $\partial M \subset M$ is homotopic to a map $\partial M \rightarrow L^{(n-1)} = M^{(n-1)}$. Similarly if W is a simply-connected ∂ -spherical oriented pseudo-bordism between M_0 and M_1 , any maps $\partial M_0 \rightarrow M_0^{(n-1)}$ and $\partial M_1 \rightarrow M_1^{(n-1)}$ homotopic to the inclusions $\partial M_0 \subset M_0$ and $\partial M_1 \subset M_1$ are homotopic with values in $W^{(n)}$. \square

Proof of Theorem 4.4. By Proposition 3.5(a), X is an inverse limit of $(m-1)$ -connected polyhedra P_i . Obviously, the Hurewicz homomorphisms fit into the long exact sequence

$$\cdots \rightarrow \Gamma'_n(X) \rightarrow \pi_n(X) \rightarrow H_n(X) \rightarrow \Gamma'_{n-1}(X) \rightarrow \cdots$$

where $\Gamma'_n(X)$ is the group of the classes of proper singular oriented pseudo-bordism, restricting to proper homotopy on the boundary, of proper singular oriented $(n+2)$ -pseudo-manifolds with boundary $S^n \times [0, \infty)$ in $P_{[0, \infty)}$. Similarly to the proof of Theorem 3.1(b), there is a short exact sequence

$$0 \rightarrow \lim_{\leftarrow} {}^1\Gamma'_{n+1}(P_i) \rightarrow \Gamma'_n(X) \rightarrow \lim_{\leftarrow} \Gamma'_n(P_i) \rightarrow 0$$

where $\Gamma'_n(P_i)$ is defined in the proof of Lemma 4.5 and coincides with $\Gamma_n(P_i)$ as shown in that proof. Since each P_i is $(m-1)$ -connected, $\Gamma_n(P_i) = 0$ for $n \leq m$.

From the hypothesis and Theorem 3.1(b), $\varprojlim^1 \pi_m(P_i) = 0$, so by Lemma 3.3, the $\pi_m(P_i)$ are Mittag-Leffler. Then by Lemma 4.6 below, the $\Gamma_{m+1}(P_i)$ are Mittag-Leffler and therefore $\varprojlim^1 \Gamma_{m+1}(P_i) = 0$. Thus $\Gamma'_n(X) = 0$ for $n \leq m$, and the assertion follows. \square

Lemma 4.6. *Let P_i be an inverse sequence of $(m-1)$ -connected polyhedra, where $m > 1$. If the $\pi_m(P_i)$ are Mittag-Leffler, then so are the $\Gamma_{m+1}(P_i)$.*

Proof. Suppose that $\pi_m(P_k)$ maps onto the image of $\pi_m(P_j)$ in $\pi_m(P_i)$, $k > j$. By cellular approximation, $\pi_m(P_k^{(m)})$ maps onto the image of $\pi_m(P_j^{(m)})$ in $\pi_m(P_i^{(m+1)})$. Pick a homotopy equivalence $f: \bigvee S^m \rightarrow P_j^{(m)}$, and let $\alpha_1, \dots, \alpha_r \in \pi_m(\bigvee S^m)$ be the basis given by the factors of the wedge. Then there exist $\beta_1, \dots, \beta_r \in \pi_m(P_k^{(m)})$ mapping onto the images of $\alpha_1, \dots, \alpha_r$ in $\pi_m(P_i^{(m+1)})$. Combining together their representatives yields a map $g: \bigvee S^m \rightarrow P_k^{(m)}$ such that the two compositions $h_1: \bigvee S^m \xrightarrow{g} P_k^{(m)} \rightarrow P_i^{(m+1)}$ and $h_2: \bigvee S^m \xrightarrow{f} P_j^{(m)} \rightarrow P_i^{(m+1)}$ induce the same map on π_m . Then h_1 and h_2 are homotopic. Since f is invertible up to homotopy, $P_j^{(m)} \rightarrow P_i^{(m+1)}$ factors up to homotopy through $P_k^{(m)} \rightarrow P_i^{(m+1)}$. Hence $\pi_{m+1}(P_k^{(m)})$ maps onto the image of $\pi_{m+1}(P_j^{(m)})$ in $\pi_{m+1}(P_i^{(m+1)})$. Therefore $\Gamma_{m+1}(P_k)$ maps onto the image of $\Gamma_{m+1}(P_j)$ in $\Gamma_{m+1}(P_i)$. \square

Remark. The following versions of Theorem 4.4 are found in the literature.

Christie [Ch; 5.12] proved that if X is UV_{m-1} , where $m > 1$, the Hurewicz homomorphism $\tilde{\pi}_m(X) \rightarrow \tilde{H}_m(X)$ is an isomorphism. This was rediscovered by K. Kuperberg (1972) and partially by T. Porter (1973). Artin and Mazur [AM; 4.5] prove a pro-group version of this result, which additionally implies that under the same hypothesis, $\pi_{m-1}(X) \rightarrow H_{m-1}(X)$ is also an isomorphism; an easier proof is found in [DS1]. Both these results follow immediately from Proposition 3.5(a) and the classical Hurewicz Theorem.

Kodama and Koyama [KK], [Ko4] prove that if X is UV_{m-1} , where $m > 2$, the Hurewicz homomorphism $\pi_m(X) \rightarrow H_m(X)$ is an isomorphism. This result follows by the proof of Theorem 4.4, if instead of Lemma 4.6 one uses that each $\Gamma_{m+1}(P_i)$ is finite for $m > 2$. (The latter holds since $\pi_{m+1}(\bigvee S^m)$ is finite for $m > 2$, which in turn is easy to see using the Pontryagin–Thom construction.) As noted in [KK], this result does not extend to the case $m = 2$, by considering the suspension of the p -adic solenoid for some p .

Finally, as noted by Kuperberg [Ku2], $\tilde{\pi}_n(X) = 0$ for all $n < 3$ does not imply $\tilde{\pi}_3(X) \simeq \tilde{H}_3(X)$. Indeed, consider the double suspension of a p -adic solenoid with p odd. The assertion follows since an odd degree self-map of S^3 induces the identity on $\pi_4(S^3) \simeq \mathbb{Z}/2$, which in turn is easy to see using the Pontryagin–Thom construction.

Proposition 4.7. *If $\pi_0(X) = 0$, the Hurewicz homomorphism $\tilde{\pi}_1(X) \rightarrow \tilde{H}_1(X)$ is surjective with kernel equal to the closure of the commutator subgroup.*

The first assertion was proved by Mardešić–Ungar, while stated by Dydak [D1]. It does not generalize to connected compacta X [D1]. If E is the Hawaiian earring (see Example 5.6), the commutator subgroup of $\tilde{\pi}_1(E)$ is not closed in $\tilde{\pi}_1(E)$ [D1]. An erroneous version of Proposition 4.7 which did not address these two counterexamples to it, was stated without proof in [Ch].

Proof. Suppose that X is the inverse limit of polyhedra P_i . Let us write $\pi_1(P_i) = G_i$. Each $G_i \rightarrow H_1(P_i)$ is onto with kernel G'_i . By Theorem 3.1(b) and Lemma 3.3, the G_i are Mittag-Leffler. Then the commutator subgroup of the stable image of G_j , $j > i$, in G_i is the stable image of G'_j , $j > i$, hence the G'_i are also Mittag-Leffler. Hence $f: \varprojlim G_i \rightarrow \varprojlim H_1(P_i)$ is onto. Since f is a continuous homomorphism to an abelian group, its kernel contains the closure of the commutator subgroup of $\varprojlim G_i$. Conversely, if a thread $(g_0, g_1, \dots) \in \ker f$, then each $g_i \in G'_i$, moreover g_i lies in the stable image of G_j , $j > i$. Then $g_i = [h_{i1}, h_{i1'}] \dots [h_{ir}, h_{ir'}]$, where each h_{ik} lies in the stable image of G_j . Let $\hat{g}_i = [h_1, h'_1] \dots [h_r, h'_r]$ for some preimages h_k of h_{ik} in $\varprojlim G_j$, then $\hat{g}_0, \hat{g}_1, \dots$ converges to the thread (g_0, g_1, \dots) . \square

As a final remark, if X is a connected compactum, $\pi_0(X) \rightarrow \tilde{H}_0(X)$ is an epimorphism by the classical Hurewicz Theorem and Theorem 3.1(b).

5. SOME EXAMPLES

The purpose of this section is twofold: to illustrate the considerations of the preceding sections and to motivate those of the subsequent ones.

Example 5.1 (null-sequence). Let \mathbb{N}^+ be the one-point compactification of the countable discrete space \mathbb{N} . Theorem 4.1(iii) implies $H_0(\mathbb{N}^+) \simeq \varprojlim H_0(\{1, \dots, i\}) \simeq \prod \mathbb{Z}$ (countable product). On the other hand, since a singular 0-cycle is supported by only finitely many points, $\hat{\tau}: \hat{H}_0(\mathbb{N}^+) \rightarrow H_0(\mathbb{N}^+)$ is an injection onto the subgroup $\langle (1, 1, \dots) \rangle \oplus (\bigoplus \mathbb{Z})$ (countable sum). At the same time, it is easy to see that both $\pi_0(\mathbb{N}^+)$ and $\hat{\pi}_0(\mathbb{N}^+)$ are homeomorphic to \mathbb{N}^+ .

Example 5.2 (Cantor set). Let $\mathbb{Z}_p = \varprojlim (\dots \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p)$ be the topological group of p -adic integers, where each \mathbb{Z}/p^n is endowed with the discrete topology. We have $\pi_0(\mathbb{Z}_p) \cong \varprojlim \pi_0(\mathbb{Z}/p^i) \cong \mathbb{Z}_p$ (as spaces). Clearly $\hat{\pi}_0(\mathbb{Z}_p) \xrightarrow{\hat{\tau}} \pi_0(\mathbb{Z}_p)$ is a homeomorphism. On the other hand, it is easy to see that $\hat{H}_0(\mathbb{Z}_p)$ is isomorphic, as a topological group, to the additive group of the group ring $\mathbb{Z}[\mathbb{Z}_p]$. At the same time, a single element of $H_0(\mathbb{Z}_p) \simeq \varprojlim H_0(\mathbb{Z}/p^i) \simeq \varprojlim \mathbb{Z}[\mathbb{Z}/p^i]$ may well “involve” all points of \mathbb{Z}_p . For instance, the elements $\sum_{i=0}^{n-1} m_i t^i \in \mathbb{Z}[\mathbb{Z}/2^n]$ where each $m_{\alpha_0+2\alpha_1+\dots+2^{n-1}\alpha_{n-1}} = (-1)^{\alpha_0} + (-1)^{\alpha_1} \cdot 2 + \dots + (-1)^{\alpha_n} \cdot 2^{n-1} \neq 0$ (here each $\alpha_i = 0$ or 1) form a thread in $\varprojlim \mathbb{Z}[\mathbb{Z}/2^n] \simeq H_0(\mathbb{Z}_2)$.

Example 5.3 (p -adic solenoid). Let Σ_p be the p -adic solenoid, i.e. the mapping torus of the homeomorphism $h: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, $h(a) = a + 1$. From the exact sequence of the pair (Σ_p, \mathbb{Z}_p) we get $0 \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}_p \rightarrow \hat{\pi}_0(\Sigma_p) \rightarrow 0$, whence $\hat{\pi}_0(\Sigma_p) \cong \mathbb{Z}_p/\mathbb{Z}$ (as spaces). Since the same argument also applies to $\pi_0(\Sigma_p)$ (see Theorem 3.1(a)), $\hat{\pi}_0(\Sigma_p) \xrightarrow{\hat{\tau}} \pi_0(\Sigma_p)$ is a bijection. The Steenrod group $\pi_0(\Sigma_p)$ can be alternatively computed using the Milnor exact sequence of Theorem 3.1(b): $\varprojlim \pi_0(S^1) = pt$, so $\pi_0(\Sigma_p) \simeq \varprojlim^1 \pi_1(S^1) \simeq \mathbb{Z}_p/\mathbb{Z}$ (see Example 3.2).

As for the homology, $\hat{H}_0(\Sigma_p) \xrightarrow{\hat{\tau}} H_0(\Sigma_p)$ has a nontrivial kernel, contrary to an assertion in [Fe3; p. 152]. We have $\tilde{H}_0(\Sigma_p) = \varprojlim^1 (\dots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}) \simeq \mathbb{Z}_p/\mathbb{Z}$. Since the composition $\tilde{H}_0(pt) \simeq H_0(pt) \rightarrow H_0(\Sigma_p) \rightarrow \tilde{H}_0(\Sigma_p) \simeq \tilde{H}_0(pt)$ is the identity,

$\check{H}_0(\Sigma_p)$ splits off as a direct summand in $H_0(\Sigma_p)$. Hence $H_0(\Sigma_p) \simeq \mathbb{Z} \oplus (\mathbb{Z}_p/\mathbb{Z})$. On the other hand, from the exact sequence of the pair (Σ_p, \mathbb{Z}_p) we obtain $\check{H}_0(\Sigma_p) \simeq \mathbb{Z}[\mathbb{Z}_p/\mathbb{Z}]$. (A computation of $H_0(\Sigma_p)$ from the pair (Σ_p, \mathbb{Z}_p) is also possible, but far less trivial; see [M2; Appendix].)

Remark. In fact, it is well-known that if X is an inverse limit of fibrations (or even Steenrod fibrations) between polyhedra, $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}} \pi_n(X)$ is a bijection for all n [Vo], [BK; IX.3.1], [Fe2; 5.5(i)].

Example 5.4 (cluster of solenoids). Let X be the cluster of solenoids, i.e. the one-point compactification of the countable disjoint union of copies of $\Sigma_p \setminus pt$, with compactifying point b . We claim that $\hat{\tau}: \hat{H}_0(X) \rightarrow H_0(X)$ is not surjective. By the Cluster Axiom (see Theorem 4.1), $H_0(X, b)$ is naturally isomorphic to the countable product of copies of $\check{H}_0(\Sigma_p) \simeq \mathbb{Z}_p/\mathbb{Z}$. A singular 0-cycle has support in some finite sub-wedge $\Sigma_p \vee \cdots \vee \Sigma_p$ of the cluster. Hence $\hat{H}_0(X, b)$ maps into (in fact, onto) the countable sum $\bigoplus \mathbb{Z}_p/\mathbb{Z} \subset \prod \mathbb{Z}_p/\mathbb{Z}$.

Example 5.5 (wedge of solenoids). We claim that $\hat{\tau}: \hat{\pi}_0(X) \rightarrow \pi_0(X)$ is not surjective, where $X = \Sigma_p \vee \Sigma_p$. Since the Hurewicz homomorphism $h: \pi_0(X) \rightarrow \check{H}_0(X)$ is surjective (see the concluding remark in §4), it sends some element of $\pi_0(X)$ onto $(\alpha + \mathbb{Z}, \alpha + \mathbb{Z}) \in (\mathbb{Z}_p/\mathbb{Z}) \oplus (\mathbb{Z}_p/\mathbb{Z})$ for some $\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}$. On the other hand, a representative of any element of $\hat{\pi}_0(X, b)$, where b is the basepoint of the wedge, has support in one of the two factors of the wedge. Hence $\hat{\pi}_0(X, b)$ maps into (in fact, onto) the subset $[(\mathbb{Z}_p/\mathbb{Z}) \oplus 0] \cup [0 \oplus (\mathbb{Z}_p/\mathbb{Z})]$ of $H_0(X, b)$.

Remark. Every 1-dimensional Steenrod connected compactum has the shape of either the Hawaiian earring or a finite wedge of circles or a point (A. Trybulec; see [DS1; 7.3.3]).

Example 5.6 (Hawaiian earring). Let E be the Hawaiian earring, that is the one-point compactification of $\mathbb{R} \times \mathbb{N}$. Then $\pi_1(E) \simeq \varprojlim \pi_1(\bigvee_{i=1}^r S^1) \simeq \varprojlim F_r$, where F_r denotes the free group $\langle x_1, \dots, x_r \mid \rangle$. On the other hand, if $f: (S^1, pt) \rightarrow (E, x)$ is a map, where x is the compactifying point, we may assume that f is transverse to each $x_n := (0, n) \in \mathbb{R} \times \mathbb{N}$. The preimage of all these points is a framed 0-submanifold of $S^1 \setminus x$ with components indexed by the letters $x_1^{\pm 1}, x_2^{\pm 1} \dots$ so that each x_i occurs only finitely many times. Similarly to the usual Pontryagin–Thom construction, it is easily verified that $\hat{\pi}_1(E)$ is isomorphic to the bordism group of such framed manifolds. It follows that $\hat{\pi}_1(E)$ is isomorphic to the subgroup L of $\varprojlim F_r$ consisting of all threads (\dots, g_2, g_1) such that when g_i 's are written as reduced words w_i in their free groups, each x_j occurs only boundedly many times (i.e. at most $n(j)$ times for some function $n(j)$) in each w_i . Thus $\hat{\pi}_1(E) \xrightarrow{\hat{\tau}} \pi_1(E)$ has nontrivial cokernel.

In contrast, the abelianization $\hat{H}_1(E)$ of L clearly maps onto $H_1(E) \simeq \prod_{i=1}^{\infty} \mathbb{Z}$. It turns out that the kernel of this surjection splits off as a direct summand in $\hat{H}_1(E)$ and is isomorphic to $\bigoplus_{\mathfrak{c}} \mathbb{Q} \oplus \bigoplus_p A_p$, where \mathfrak{c} is the cardinality of the continuum, p runs over all primes, and A_p is the p -adic completion of $\bigoplus_{\mathfrak{c}} \mathbb{Z}_p$ [EK2]. This is found by a direct computation of the abelianization of L .

Example 5.7 (Hawaiian snail). Let E^n be the n -dimensional Hawaiian earring, now viewed as the one-point compactification of $\mathbb{R}^n \times \mathbb{Z}$. Let $\sigma_n: E^n \rightarrow E^n$ be

the homeomorphism given by the shift $\mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i + 1$. Let X be the mapping torus of σ_n , where $n \geq 2$. From the Milnor exact sequence it is easy to see that $\pi_n(X) \simeq H_n(X) = 0$. On the other hand, if \tilde{X} is the universal cover of X , it is easy to see that $\hat{\pi}_n(\tilde{X})$ contains $\mathbb{Z}[\mathbb{Z}]$. Hence $\hat{\pi}_n(X)$ does not inject into $\pi_n(X)$.

To compute $\hat{H}_n(X)$, we first note that by the Hurewicz Theorem (which is proved in the required generality in [Sp], for instance), $\hat{H}_n(E^n) \simeq \hat{\pi}_n(E^n)$ for $n > 1$; and by the Pontryagin–Thom construction (see Example 5.6 or the proof of Theorem 1.1; compare [EK3]), $\hat{\pi}_n(E^n) \simeq \prod_{i=1}^{\infty} \mathbb{Z}$ for $n > 1$. Then the exact sequence of the pair (X, E^n) yields $\prod_{i=1}^{\infty} \mathbb{Z} \xrightarrow{\partial} \prod_{i=1}^{\infty} \mathbb{Z} \rightarrow \hat{H}_n(X) \rightarrow 0$, where the boundary map ∂ is given by $1 - t: \mathbb{Z}[[t^{\pm 1}]] \rightarrow \mathbb{Z}[[t^{\pm 1}]]$. But this is an isomorphism (its inverse is given by $1 + t + t^2 + \dots$), so $\hat{H}_n(X) = 0$.

Example 5.8 (Hawaiian snail with partitions). Let X be the mapping torus of σ_{n-1} from the previous example, where $n \geq 3$. Consider an LC_{n-1} compactum Y obtained from X by attaching small n -disks D_1, D_2, \dots . We will show that $\tilde{\tau}\hat{\tau}: \hat{\pi}_n(Y) \rightarrow \tilde{\pi}_n(Y)$ is not surjective. (Note that Theorem 6.5 below implies that $\tilde{\tau}\hat{\tau}: \hat{\pi}_n(Y^+) \rightarrow \tilde{\pi}_n(Y^+)$ is surjective, where Y^+ is obtained by attaching a 2-disk to Y along the orbit of the compactifying point.)

Clearly $H_n(X) \simeq \check{H}_n(X) \simeq \mathbb{Z}$ and $\pi_n(X) \simeq \tilde{\pi}_n(X) \simeq \mathbb{Z}[\mathbb{Z}]$. Pick some $a \in \pi_n(X)$ whose image in $H_n(X)$ is non-trivial, and let $b \in \pi_n(Y)$ be the image of a . Suppose that $b = \hat{\tau}([f])$ for some spheroid $f: S^n \rightarrow Y$. Then f lifts to the universal cover Y_{∞} . Since S^n is compact, there exists a finite number r such that the image of the lift of f is contained in a union of r translates of a fundamental domain F_0 of the action of \mathbb{Z} on Y_{∞} . Let $f_{r+1}: S^n \rightarrow Y_{r+1}$ be a lift of f into the $(r+1)$ -fold cyclic cover of Y . Then $f_{r+1}(S^n)$ is disjoint from some translate F_i of the fundamental domain of $\mathbb{Z}/(r+1)$.

By Theorem 3.15(b), $\pi_n(X_{r+1}) \rightarrow \pi_n(X)$ is an isomorphism, where X_{r+1} is the $(r+1)$ -fold cyclic cover of X . So a is the image of some $a_{r+1} \in \pi_n(X_{r+1})$. The image $b_{r+1} \in \pi_n(Y_{r+1})$ of a_{r+1} is represented by a translate of f_{r+1} , so b_{r+1} maps trivially to $\pi_n(Y_{r+1}, \text{Cl}(Y_{r+1} \setminus F_j))$ for some copy F_j of the fundamental domain. In particular, it maps trivially to $H_n(Y_{r+1}, \text{Cl}(Y_{r+1} \setminus F_j))$. Let $F'_j = F_j \cap X_{r+1}$ be the corresponding fundamental domain of the covering map $X_{r+1} \rightarrow X$. Since Y_{r+1} is n -dimensional, $H_n(X_{r+1}, \text{Cl}(X_{r+1} \setminus F'_j)) \rightarrow H_n(Y_{r+1}, \text{Cl}(Y_{r+1} \setminus F_j))$ is injective. Since $\text{Cl}(X_{r+1} \setminus F'_j)$ deformation retracts onto a copy of the $(n-1)$ -dimensional compactum E^{n-1} , $H_n(X_{r+1}) \rightarrow H_n(X_{r+1}, \text{Cl}(X_{r+1} \setminus F'_j))$ is injective. Thus a_{r+1} maps trivially to $H_n(X_{r+1})$. But then a maps trivially to $H_n(X)$, which contradicts our choice of a .

Remark. Another LC_{n-1} compactum Y such that $\tilde{\tau}\hat{\tau}: \hat{\pi}_n(Y) \rightarrow \tilde{\pi}_n(Y)$ is not surjective is $Y = S^1 \vee E^n$, where the basepoint of the Hawaiian earring E^n is taken at its compactifying point. This was observed by Dydak [D2; Example 8.10] and Zdravkovska [Zd], of which the present author was unaware when writing up Example 5.8. Zdravkovska also shows that $\hat{\pi}_n(X) \rightarrow \pi_n(X)$ is not surjective for any compactum X , Steenrod homotopy equivalent to Y . In contrast, Ferry proved that every UV_1 compactum is Steenrod homotopy equivalent to a compactum X such that $\hat{\tau}: [Z, X]^{\wedge} \rightarrow [Z, X]$ is a bijection for every finite-dimensional compactum Z [Fe2]. His construction always produces an infinite dimensional X .

6. COMPARISON OF THEORIES

We recall that a compactum X is *locally n -connected* (\hat{LC}_n), if every neighborhood U of every $x \in X$ contains a neighborhood V of x such that $\hat{\pi}_i(V) \rightarrow \hat{\pi}_i(U)$ is trivial for all $i \leq n$. We say that X is *semi- \hat{LC}_n* if it is \hat{LC}_{n-1} and every $x \in X$ has a neighborhood V such that $\hat{\pi}_n(V) \rightarrow \hat{\pi}_n(X)$ is trivial.

The singular group $\hat{\pi}_n(X)$ can be topologized similarly to $\pi_n(X)$, that is by declaring a subset to be open if it is the preimage of an open subset of $\check{\pi}_n(X)$; or equivalently if it is a point-inverse of the homomorphism induced by some map of X into a polyhedron. Note that $\hat{\pi}_n(X)$ and $\pi_n(X)$ are in general non-Hausdorff, whereas $\check{\pi}_n(X)$ is metrizable.

Theorem 6.1. *Fix some $n \geq 0$ and let X be a compactum.*

(a) *The following conditions are equivalent:*

- \hat{LC}_n : X is locally n -connected with respect to $\hat{\pi}_i(\cdot)$;
- LC_n : X is locally n -connected with respect to $\pi_i(\cdot)$;
- \check{LC}_n : X is locally n -connected with respect to $\check{\pi}_i(\cdot)$.

(b) *If X is LC_1 or $n = 0$, the following conditions are also equivalent to the preceding ones:*

- $HL\hat{C}_n$: X is locally n -connected with respect to $\hat{H}_i(\cdot)$;
- HLC_n : X is locally n -connected with respect to $H_i(\cdot)$;
- $HL\check{C}_n$: X is locally n -connected with respect to $\check{H}_i(\cdot)$.

(c) *If X is \hat{LC}_n , then $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}} \pi_n(X)$ and $\pi_n(X) \xrightarrow{\check{\tau}} \check{\pi}_n(X)$ are bijective.*

(d) *If X is semi- \hat{LC}_n , then $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}\hat{\tau}} \check{\pi}_n(X)$ is bijective.*

(e) *If X is \hat{LC}_{n-1} , then $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}\hat{\tau}} \check{\pi}_n(X)$ is onto a dense subset.*

(f) *X is semi- \hat{LC}_n iff X is \hat{LC}_{n-1} and $\hat{\pi}_n(X)$ is discrete.*

(g) *X is semi- LC_n and $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}} \pi_n(X)$ is onto iff X is LC_{n-1} and $\pi_n(X)$ is discrete.*

(h) *X is semi- \check{LC}_n iff X is \check{LC}_{n-1} and $\check{\pi}_n(X)$ is discrete.*

Most of Theorem 6.1 is contained in the literature. Hurewicz proved that X is \hat{LC}_1 and $HL\hat{C}_n$ if and only if it is \hat{LC}_n [Hu]. Borsuk proved an equivalent of the second assertion of (c) [Bo]. Part (d) was proved by Kuperberg [Ku1] and under the \hat{LC}_n hypothesis also by Bogatyĭ [Bog]. Dydak proved part (e) and that if X is \hat{LC}_n , then $\check{\pi}_n(X)$ is discrete [D2; Theorem 8.7]. He obtains these along with both assertions of (c) by a modification of the proof of his pro-group version of the Smale theorem (see Theorem 3.15(e)); this local-global approach, similar in spirit to the Zeeman spectral sequence, provides a substantial alternative to the methods used in the present section.

For clarity we suppress all basepoints in the proof and leave it to the reader to verify that this does not cause confusion. The proofs of all parts depend on

Ferry's Construction [Fe2; p. 381]. We represent X as the limit of an inverse sequence P of polyhedra and PL maps, and use the notation introduced in the beginning of §2. Let us triangulate $P_{[0,\infty)}$ so that the diameters of simplices tend to zero as they approach X . Assuming that X is \hat{LC}_n , it follows by induction on n

that for each $\varepsilon > 0$ there exists a k such that $Y := X \cup P_{[k,\infty)}^{(n+1)}$ retracts onto X by a retraction $r_k^{(n+1)}$ that is ε -close to the identity. Since $r_k^{(n+1)}$ is close to the identity and $P_{[0,\infty]}$ is $\hat{L}\hat{C}_\infty$, for each l there exists a k such that $r_k^{(n+1)}$ is homotopic to the identity by a homotopy $Y \times I \rightarrow P_{[l,\infty]}$ keeping X fixed. If X is only semi- $\hat{L}\hat{C}_n$, the restriction of $r_k^{(n)}$ to $P_{[k,\infty)}^{(n)}$ still extends to a map $P_{[k,\infty)}^{(n+1)} \rightarrow X$, which however is no longer close to the identity on top-dimensional simplices.

Proof of Theorem 6.1. (e). Let us represent the given $\alpha \in \tilde{\pi}_n(X)$ by a Steenrod spheroid $\Phi: S^n \times [0, \infty) \rightarrow P_{[0,\infty)}$. We may assume that each $\varphi_i = \Phi|_{S^n \times \{i\}}: S^n \rightarrow P_i$ goes into the n -skeleton. If X is $\hat{L}\hat{C}_{n-1}$, for each l there exists a k such that $r_k^{(n)}$ is homotopic to the identity within $P_{[l,\infty]}$, and therefore $\psi_k := r_k^{(n)}\varphi_k: S^n \rightarrow X$ is homotopic to φ_k within $P_{[l,\infty]}$. The strong deformation retraction of $P_{[l,\infty]}$ onto P_l carries this homotopy onto a homotopy between $p_l^\infty\psi_k$ and $p_l^k\varphi_k$. It also carries $\Phi|_{S^n \times [l,k]}$ onto a homotopy between $p_l^k\varphi_k$ and φ_l . Hence $[\varphi_l] \in \pi_n(P_l)$ is the image of $[\psi_k] \in \hat{\pi}_n(X)$. It follows that $\tilde{\tau}\hat{\tau}([\psi_k]) \in \tilde{\pi}_n(X)$ converge to α . \square

(d): Surjectivity. We use the notation in the proof of (e). If X is semi- $\hat{L}\hat{C}_n$, each ψ_k is homotopic to ψ_{k+1} via $r_k^{(n+1)}\Phi|_{S^n \times [k,k+1]}$. So the $\tilde{\tau}\hat{\tau}([\psi_k])$ are all equal. Since $\tilde{\pi}_n(X)$ is Hausdorff, they also equal their limit α . \square

(d): Injectivity [Fe2; p. 381]. We will prove that $\hat{\pi}_n(X) \xrightarrow{\tilde{\tau}\hat{\tau}} \tilde{\pi}_n(X) \xrightarrow{p_k^\infty} \pi_n(P_k)$ is injective. Given a singular spheroid $\varphi: S^n \rightarrow X$ in the kernel of this composition, it bounds a null-homotopy $\hat{\varphi}: B^{n+1} \rightarrow P_{[l,\infty]}$. By the cellular approximation theorem, we may assume that its image is in the $(n+1)$ -skeleton (apart from ∂B^{n+1} , which is mapped into X). Then $r_k^{(n+1)}\hat{\varphi}$ is a null-homotopy of φ . \square

(c). This follows either from (d), Theorem 3.1(b) and part (1) of the following lemma, or from the injectivity in (d), Theorem 3.1(b) and part (2) of the following lemma.

Lemma 6.2. *Fix some $n \geq 0$ and let X be a compactum.*

- (1) (Borsuk [Bo]) *If X is $\hat{L}\hat{C}_n$, then $\pi_n(X) \xrightarrow{\tilde{\tau}} \tilde{\pi}_n(X)$ is injective.*
- (2) *If X is $\hat{L}\hat{C}_n$, then $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}} \pi_n(X)$ is surjective.*
- (3) (Hurewicz [Hu]) *If X is $\hat{L}\hat{C}_{n-1}$ and $\hat{L}\hat{C}_n$, then $\hat{\pi}_n(X) \xrightarrow{\tilde{\tau}\hat{\tau}} \tilde{\pi}_n(X)$ is injective.*

Proof. (1) [Fe2; p. 381]. Given a map $\varphi: S^{n+1} \rightarrow P_k$, its image may be assumed to lie in the $(n+1)$ -skeleton, and then $r_k^{(n+1)}\varphi: S^{n+1} \rightarrow X$ is homotopic to φ with values in $P_{[l,\infty)}$. Thus $\hat{\pi}_{n+1}(X)$ maps onto the image of $\pi_{n+1}(P_k)$ in $\pi_{n+1}(P_l)$. It follows that the $\pi_{n+1}(P_i)$ are Mittag-Leffler, and the assertion follows from Theorem 3.1(b). \square

(2). Given a Steenrod spheroid $\chi: S^n \times [0, \infty) \rightarrow P_{[0,\infty)}$ with image in the $(n+1)$ -skeleton, each $t \in [k, \infty)$ yields a singular spheroid $\varphi_t = r_k^{(n+1)}\chi|_{S^n \times \{t\}}: S^n \rightarrow X$. The image of $[\varphi_k]$ in $\pi_n(X)$ is represented by $\Phi_k^+: S^n \times [0, \infty) \rightarrow P_{[0,\infty)}$ defined by $\Phi_k^+(x, t) = \Pi_t(\varphi_k(x))$ (the homotopy Π_t is defined in the beginning of §2). The family of maps $\Phi_t: S^n \times [t, \infty) \rightarrow P_{[t,\infty)}$, defined by $\Phi_t(x, s) = \Pi_s(\varphi_t(x))$, yields a proper homotopy between Φ_k and the map $\Psi: S^n \times [k, \infty) \rightarrow P_{[k,\infty)}$ defined by $\Psi(x, t) = \Phi_t(x, t)$. On the other hand, the same family Φ_t yields a homotopy between Ψ and $r_k^{(n+1)}\chi|_{S^n \times [k,\infty)}$. The latter is homotopic to $\chi|_{S^n \times [k,\infty)}$ since $r_k^{(n+1)}$

is homotopic to the identity. The resulting homotopy $h: S^n \times [k, \infty) \times I \rightarrow P_{[0, \infty]}$ between $\chi|_{S^n \times [k, \infty)}$ and Ψ is such that $h^{-1}(C)$ is compact for every compact $C \subset P_{[0, \infty)}$. Therefore it can be pushed in so as to become a proper homotopy within $P_{[k, \infty)}$. The resulting proper homotopy between $\chi|_{S^n \times [k, \infty)}$ and Φ_k can be extended to a proper homotopy between χ and Φ_k^+ using Borsuk's lemma. \square

(3) [DS3; §3]. Let $\varphi: S^n \rightarrow X$ be a singular spheroid that is Čech trivial. Arguing as in the proof of the injectivity in 6.1(d), we obtain that it bounds a null-homotopy $\Phi: B^{n+1} \rightarrow P_{[k, \infty]}$ for any k given in advance. In particular, given an $\varepsilon > 0$, we may choose k so that $r_k^{(n)}$ is ε -close to the identity and the simplices of the triangulation of $P_{[k, \infty)}$ have diameters $< \varepsilon$. By cellular approximation, we may assume that Φ sends the n -skeleton $T^{(n)}$ of some triangulation T of the interior of B^{n+1} into the n -skeleton of $P_{[k, \infty)}$, and moreover that $\Phi(\partial\sigma)$ has diameter $< 3\varepsilon$ for every $(n+1)$ -simplex σ of T . Then the image of the restriction $\varphi_\sigma: \partial\sigma \rightarrow X$ of $r_k^{(n)}\Phi: T^{(n)} \rightarrow X$ has diameter $< 5\varepsilon$. Moreover, since $P_{[k, \infty]}$ is LC_∞ , given any $\delta > 0$, we may choose ε so that each φ_σ is δ -null-homotopic in $P_{[k, \infty]}$. Since X is LC_n , φ_σ bounds a δ -null-homotopy $\Phi_\sigma: \sigma \rightarrow P_{[k_1, \infty]}$ for any k_1 given in advance.

We may now repeat the entire process, with a smaller ε , separately for each $\varphi_\sigma: S^n \rightarrow X$, where σ runs over all $(n+1)$ -simplices of T . The construction converges to a map f into X defined on the union U of the n -skeleta $T_i^{(n)}$ of a sequence of successive subdivisions T_i of T such that each $(n+1)$ -simplex Δ of T_i has diameter $< 2^{-i}$ and $f(\partial\Delta)$ has diameter $< 2^{-i}$. Since U is a dense subset of B^{n+1} , this also defines f on the whole of B^{n+1} , and this extension is easily seen to be well-defined and continuous. \square

Proof of Theorem 6.1, continued. (a). If X is LC_n , then by a local version¹⁹ of Lemma 6.2 (2), X is LC_n . (Note that an alternative proof of (2) can be obtained from (d) and (1).) If X is LC_n , then it is LC_n since $\tilde{\tau}$ is always an epimorphism. Finally, to prove that LC_n implies LC_n , we may assume, arguing by induction, that X is LC_{n-1} . Then the assertion follows from a local version of Lemma 6.2 (3). \square

(h), \Rightarrow . If an element of $\pi_n(X)$ maps trivially to $\pi_n(P_k)$, it can be represented by a level-preserving Steenrod spheroid $\chi: S^n \times [0, \infty) \rightarrow P_{[0, \infty)}$ whose restriction to $S^n \times \{k\}$ bounds a null-homotopy $\varphi: D^{n+1} \rightarrow P_k$. Let $\psi: \mathbb{R}^{n+1} \rightarrow P_{[k, \infty)}$ be the proper map obtained by combining φ with the restriction of χ to $S^n \times [k, \infty)$. We may assume that ψ is cellular with respect to some triangulation of \mathbb{R}^{n+1} . Since X is LC_{n-1} , the retraction $r_k^{(n)}: Y \rightarrow X$ is homotopic to the identity by a homotopy $h: Y \times I \rightarrow P_{[l, \infty]}$ keeping X fixed. If Δ is an $(n+1)$ -simplex of \mathbb{R}^{n+1} , let $\eta: (\partial\Delta) \times I \rightarrow P_{[l, \infty]}$ be the restriction of h to $\psi(\partial\Delta) \times I$. By using the homotopy Π_t from the beginning of §2, we may assume that η restricts to a proper map $(\partial\Delta) \times [0, 1] \rightarrow P_{[l, \infty)}$. Since X is semi- LC_n , this Steenrod spheroid is Čech trivial as long as l is large enough. Since Δ is arbitrary, it follows that χ is also Čech trivial. \square

¹⁹By “a local version of \mathfrak{X} ”, we mean “an appropriate straightforward generalization of \mathfrak{X} , applied to closed neighborhoods of points”. For instance, right now we are dealing with the following generalization of Lemma 6.2(2): if $Z \subset Y \subset X$ are compacta, where X is LC_n and Y is a neighborhood of Z in X , then the image of $\hat{\tau}: \hat{\pi}_n(Y) \rightarrow \pi_n(Y)$ contains the image of $i_*: \pi_n(Z) \rightarrow \pi_n(Y)$. In some cases, \mathfrak{X} can be applied directly by virtue of Theorem 6.11.

(h), \Leftarrow . By Lemma 3.4(b), the hypothesis implies that $\tilde{\pi}_n(X)$ injects into $\pi_n(P_k)$ for some k . On the other hand, if $U \subset X$ is so small that its image in P_k lies in the star of some vertex of P_k , the composition $\tilde{\pi}_n(U) \rightarrow \tilde{\pi}_n(X) \rightarrow \pi_n(P_k)$ is trivial. Then also $\tilde{\pi}_n(U) \rightarrow \tilde{\pi}_n(X)$ must be trivial. \square

(g), \Rightarrow . For convenience of notation, we assume that $n > 0$; the case $n = 0$ is similar. By the hypothesis, every element of $\pi_n(X)$ can be represented by a spheroid $\varphi: S^n \rightarrow X$. If the composition $S^n \xrightarrow{\varphi} X \xrightarrow{p_k^\infty} P_k$ is null-homotopic for some k , then φ bounds a disk $\psi: D^{n+1} \rightarrow P_{[k, \infty)}$. Since X is LC_{n-1} , given any $\varepsilon > 0$, the first stage of Hurewicz's Construction (from the proof Lemma 6.1(1)) represents $[\psi] \in \hat{\pi}_n(X)$ as a finite sum of the classes of compositions $(S^n, pt) \xrightarrow{p} (S^n \vee I, 0) \xrightarrow{f_i} (X, x)$ where p collapses a hemisphere of S^n onto the whisker I , sending the basepoint to the end of the whisker, and each $f_i(S^n)$ is of diameter at most ε . Since X is semi- LC_n , each of these is Steenrod null-homotopic. Thus $\hat{\tau}([\varphi]) = 0$. This proves that the composition $\pi_n(X) \xrightarrow{\hat{\tau}} \tilde{\pi}_n(X) \xrightarrow{p_k^\infty} \pi_n(P_k)$ is injective for some k , which implies the assertion. \square

(g), \Leftarrow . By the hypothesis, $\pi_n(X) \xrightarrow{\hat{\tau}} \tilde{\pi}_n(X)$ is a bijection and $\tilde{\pi}_n(X)$ is discrete. Hence by (e), $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}} \pi_n(X)$ is onto, and from (h), X is semi- LC_n . \square

(f). If X is semi- LC_n , $\tilde{\pi}_n(X)$ is discrete by the proof of the injectivity in (d). (Alternatively, by a local version of (e), taking into account that $\tilde{\pi}_n(X)$ is Hausdorff, it follows that X is semi- LC_n , and the assertion follows from (h).)

Conversely, if $\hat{\pi}_n(X)$ is discrete, then by the definition of its topology it injects into $\tilde{\pi}_n(X)$; let G_n denote the image of this injection. If $n = 0$, clearly the underlying set of $\tilde{\pi}_n(X)$ can be identified with the set of quasi-components of X (which coincides with the set of components since X is compact). Hence $G_0 = \tilde{\pi}_0(X)$. If $n > 0$, the topological group $\tilde{\pi}_n(X)$ can be endowed with the left-invariant metric defined as the inverse limit of the metrics on $\pi_n(P_i)$ where the distance between every two points equals 1. Since the induced metric on G_n is also left-invariant and its induced topology is discrete, it follows that its induced uniform structure is discrete. So every Cauchy sequence in G_n is eventually constant, thus G_n is closed in $\tilde{\pi}_n(X)$. Hence by (e), $G_n = \tilde{\pi}_n(X)$, and we conclude that $\tilde{\pi}_n(X)$ is discrete in both cases ($n = 0$ and $n > 0$). Then by (h), X is semi- LC_n and consequently also semi- LC_n . \square

(b). If X is LC_n , then $\hat{H}_n(X) \xrightarrow{\hat{\tau}} H_n(X)$ and $H_n(X) \xrightarrow{\hat{\tau}} \check{H}_n(X)$ are isomorphisms similarly to the proof of (c), using pseudo-manifolds instead of $S^n \times [0, \infty)$. If X is LC_{n-1} and HLC_n , this argument still works: the retraction $r_k^{(n+1)}$ becomes defined “up to attaching handles to the top-dimensional simplices”, that is, it is now defined on $Y' := X \cup Q_k$, where Q_k is a polyhedron obtained by replacing each top-dimensional simplex Δ^{n+1} of $P_{[k, \infty)}^{(n+1)}$ by some $(n+1)$ -pseudo-manifold with boundary $\partial\Delta^{n+1}$. This $r_k^{(n+1)}: Y' \rightarrow X$ is homotopic to the projection $Y' \rightarrow Y$ (instead of id_Y) by a homotopy $Y' \times I \rightarrow P_{[l, \infty)}$ keeping X fixed.

Obviously LC_0 is equivalent to HLC_0 ; and LC_0 is equivalent to HLC_0 . By the above, HLC_0 implies HLC_0 which in turn implies HLC_0 . Since singular 1-cycles are all spherical, LC_1 implies HLC_1 . Then by the above it also implies HLC_1 and HLC_1 . It remains to consider the case $n > 1$. Arguing by induction, we may

assume that X is $\hat{L}\hat{C}_{n-1}$. Then by the above, HLC_n implies HLC_n , which in turn implies HLC_n . On the other hand, $\hat{L}\hat{C}_n$ is equivalent to HLC_n by Lemma 6.3(a) below, and $\hat{L}\hat{C}_n$ is equivalent to HLC_n by Lemma 6.3(b) below along with Theorem 6.1(d). \square

Lemma 6.3. *Let $f: X \rightarrow Y$ be a map between compacta such that for any map g of an $(n-1)$ -polyhedron into X , the composition fg is null-homotopic. Then*

(a) $\text{im}[\hat{H}_n(X) \rightarrow \hat{H}_n(Y)]$ is contained in $\text{im}[\hat{\pi}_n(Y) \rightarrow \hat{H}_n(Y)]$, and if $n > 1$, $\ker[\hat{\pi}_n(X) \rightarrow \hat{H}_n(X)]$ is contained in $\ker[\hat{\pi}_n(X) \rightarrow \hat{\pi}_n(Y)]$.

(b) If X is $\hat{L}\hat{C}_{n-2}$ and $n > 1$, $\text{im}[\check{H}_n(X) \rightarrow \check{H}_n(Y)] \subset \text{im}[\check{\pi}_n(Y) \rightarrow \check{H}_n(Y)]$ and $\ker[\check{\pi}_n(X) \rightarrow \check{H}_n(X)] \subset \ker[\check{\pi}_n(X) \rightarrow \check{\pi}_n(Y)]$.

The second assertion of (a) is essentially the well-known “Eventual Hurewicz Theorem” of Ferry [Fe1; 3.1] (the name seems to originate from F. Quinn’s “Ends of maps — I” and has become standard). Indeed, the hypothesis of Lemma 6.3 is satisfied if f is a composition $f_{n-1} \dots f_0$ where each f_i is trivial on $\hat{\pi}_i$.

It will follow from Theorem 6.7 that the first assertion of (b) also holds for $n = 1$ if X is assumed to be $\hat{L}\hat{C}_0$.

Proof. (a). Given a singular cycle $\varphi: Z \rightarrow X$, where Z is an n -pseudo-manifold with a fixed triangulation, by the hypothesis $f\varphi$ is homotopic to a ψ that sends $Z^{(n-1)}$ to the basepoint. Thus $f_*[\varphi] = \sum[\psi_i]$, where $\psi_i: S^n \rightarrow Y$ are given by restricting ψ to the top-dimensional simplices.

Given a singular $(n+1)$ -chain $\varphi: Z \rightarrow X$ with $\partial Z = S^n$, by the hypothesis $f\varphi$ is homotopic to a ψ that sends $Z^{(n-1)}$ to the basepoint. This ψ factors through the quotient $Z' = Z/Z^{(n-1)}$, where $\pi_n(Z') \rightarrow H_n(Z')$ is an isomorphism by the classical Hurewicz Theorem. Thus $S^n = \partial Z$ is null-homotopic in Z' and therefore in Y . \square

(b). Suppose that X and Y are the inverse limits of compact polyhedra P_i and Q_i . For each l there exists a $k > l$ such that $r_k^{(n-1)}: X \cup P_{[k, \infty)}^{(n-1)} \rightarrow X$ is defined and homotopic with values in $P_{[l, \infty]}$ to the identity (see Ferry’s Construction). Then the inclusion $\varphi: K \rightarrow P_k$ of the $(n-1)$ -skeleton K of some triangulation of P_k is homotopic with values in $P_{[l, \infty]}$ to some $\psi: K \rightarrow X$. By the hypothesis, $f\psi: K \rightarrow Y$ is null-homotopic. By Lemmas 2.1 and 2.5(b₀) we may assume (after dropping some of the P_i) that f extends to a level-preserving map $F: P_{[0, \infty]} \rightarrow Q_{[0, \infty]}$. Then $F\varphi$ is null-homotopic with values in $Q_{[l, \infty]}$. Hence $F|_{P_k}: P_k \rightarrow Q_{[l, \infty]}$ factors up to homotopy through the $(n-1)$ -connected polyhedron $L := P_k/K$. By the Hurewicz Theorem, $\pi_n(L) \simeq H_n(L)$.

Consider a Steenrod spheroid $\varphi: S^n \times [0, \infty) \rightarrow P_{[0, \infty)}$ such that each $\varphi_k = \varphi|_{S^n \times k}: S^n \rightarrow P_k$ represents the trivial element of $H_n(P_k)$. Then $F\varphi_k$ represents the trivial element of $\pi_n(Q_{[l, \infty]})$. Since φ_k is homotopic with values in $P_{[l, k]}$ to φ_l , the homotopy class $[F\varphi_l]$ is also trivial in $\pi_n(Q_{[l, \infty]})$. Since $Q_{[l, \infty]}$ deformation retracts onto Q_l , the class $[F\varphi_l]$ is trivial in $\pi_n(Q_l)$.

Let $\psi: M \rightarrow P_{[0, \infty)}$ be a Steenrod cycle, and consider its level $\psi_k = \psi|_{\psi^{-1}(P_k)}$. Then $F\psi_k$ is homologous with values in $Q_{[l, \infty]}$ to a spherical cycle $\varphi_l: S^n \rightarrow Q_{[l, \infty]} \rightarrow Q_l$. Since each ψ_k is homologous to ψ_{k+1} with values in $P_{[k, k+1]}$, each φ_l is homotopic to φ_{l+1} with values in $Q_{[m+1, \infty]}$, where $l = l(m+1)$ is chosen similarly to $k = k(l)$. Then $\chi_m := \Pi_m \varphi_l$ and χ_{m+1} are level-preserving homotopic

with values in $Q_{[m,m+1]}$ (the homotopy Π_t is defined in §2). Thus χ_m are the levels of a Steenrod spheroid $S^n \times [0, \infty) \rightarrow Q_{[0,\infty)}$. On the other hand, since ψ_m is homologous to ψ_k with values in $P_{[m,k]}$, and $Q_{[m,\infty]}$ deformation retracts onto Q_m , the composition $F\psi_m$ is homologous with values in Q_m to χ_m . \square

Spheroids with trunks. An n -spheroid with trunks in a compactum X is a uniformly continuous map $\varphi: (\partial R, pt) \rightarrow (X, x_0)$, where the uniform space R is a regular ε -neighborhood of a properly embedded infinite binary tree in \mathbb{R}^{n+1} , with respect to some proper function $\varepsilon: \mathbb{R}^{n+1} \rightarrow (0, 1]$. (The word “trunk” is used here in the meaning “elephant’s trunk”.) Note that ∂R is non-uniformly homeomorphic to the complement of a tame Cantor set in S^n .

If X is the limit of an inverse sequence P of polyhedra and PL maps, similarly to the proof of Lemma 2.1, every spheroid with trunks $\varphi: \partial R \rightarrow X$ extends to a uniformly continuous map $\bar{\Phi}: R \rightarrow P_{[0,\infty]}$ that restricts to a proper map $\Phi: \text{Int}R \rightarrow P_{[0,\infty)}$, and any two such extensions are homotopic through such extensions. We shall say that φ represents $[\Phi] \in \pi_n(X)$.

Proposition 6.4. *If X is an LC_n compactum and $n \geq 0$, every element of $\pi_{n+1}(X)$ is representable by an $(n+1)$ -spheroid with trunks.*

Remark. Proposition 6.4 can be generalized as follows. If X is an LC_n compactum, every element of $\pi_{n+d}(X)$ can be represented by a uniformly continuous map $\partial R \rightarrow X$, where R is a regular ε -neighborhood in \mathbb{R}^{n+d+1} of a properly embedded mapping telescope of an inverse sequence of $(d-1)$ -polyhedra P_i (starting with $P_0 = pt$), with respect to some proper function $\varepsilon: \mathbb{R}^{n+d+1} \rightarrow (0, 1]$. In fact, P_i can be more specifically described as the dual skeleta of the successive barycentric subdivisions of some triangulation of S^{n+d} , with the obvious bonding maps. In this case, if $2d-1 \leq n+d$, then $\varprojlim P_i$ is homeomorphic to the Menger cube μ_{d-1} by Bestvina’s characterization of μ_{d-1} (see [GHW]). Thus ∂R is non-uniformly homeomorphic to the complement of a tame (by construction) copy of μ_{d-1} in S^{n+d} .

Proof. Let $\Phi: S^{n+1} \times [0, \infty) \rightarrow P_{[0,\infty)}$ be a Steenrod spheroid. Let $K^{[1]}, K^{[2]}, \dots$ be the successive barycentric subdivisions of some triangulation of S^{n+1} . (Strictly speaking, to obtain the binary rather than the $(n+1)!$ -ary tree in the end, we should have decomposed each operation of barycentric subdivision into a sequence of dichotomic subdivisions.) Let L_i be the n -skeleton of $K^{[i]}$ and let N_i be the third derived neighborhood of L_i in $K^{[i]}$. Note that $S^{n+1} \setminus \bigcup \text{Int}N_i$ is a tame Cantor set C , and there is a continuous map $\varphi: S^{n+1} \rightarrow S^{n+1}$ with $\varphi(\bigcup \text{Int}N_i) = \bigcup L_i$ and $\varphi(C) = S^{n+1}$. (It is constructed analogously to the map $[0, 1] \rightarrow [0, 1]$ that converts a ternary expansion $A = 0.\alpha_1\alpha_2$ into a binary expansion $B = 0.\beta_1\beta_2\dots$ by replacing all twos with ones and halting after the first occurrence of a one in A .) Let $N = \bigcup N_i \times [i, \infty) \subset S^{n+1} \times [0, \infty)$. On the other hand, let N' be the third derived neighborhood of $\bigcup L_i \times [i, \infty)$ in $S^{n+1} \times [0, \infty)$. Let E and E' be the exteriors of N and N' respectively in $S^{n+1} \times [0, \infty)$.

Let $f: N' \times [0, 1] \rightarrow P_{[l,\infty]}$ be the homotopy between $\Phi|_{N'}$ and $r_k^{(n+1)}\Phi|_{N'}$. Using the homotopy Π_t from the beginning of §2, we may assume that $f^{-1}(X) = N' \times \{1\}$. Let $h: S^{n+1} \times [0, \infty) \rightarrow E' \cup \partial N' \times [0, 1]$ be a homeomorphism with $h(E) = E'$. Let Φ' be defined by Φh on E and by fh on N . Then Φ' is properly homotopic to Φ (using f) and extends by continuity to a uniformly continuous map $\bar{\Phi}': S^{n+1} \times [0, \infty] \setminus C \times \{\infty\} \rightarrow P_{[0,\infty]}$. \square

Theorem 6.5. *Let X be an LC_{n-1} compactum, $n \geq 1$.*

- (a) *If $\hat{\pi}_1(X) = 1$, then $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}} \pi_n(X)$ is an epimorphism.*
- (b) *If $\hat{\pi}_1(X)$ acts trivially on $\hat{\pi}_n(X)$, then $\hat{\pi}_n(X) \xrightarrow{\hat{\tau}\hat{\tau}} \pi_n(X)$ is an epimorphism.*

Note that in the case $n \geq 2$, the hypotheses in (a) and (b) can be weakened using Theorem 3.15(b) to “ $\hat{\pi}_1(X)$ is finite” and “a finite index subgroup of $\hat{\pi}_1(X)$ acts trivially on $\hat{\pi}_n(X)$ ”, respectively.

The necessity of simple connectedness in Theorem 6.5 is shown by Example 5.8.

Proof. Let $\varphi: (\partial R, pt) \rightarrow (X, x)$ be an n -spheroid with trunks realizing the given element of $\pi_n(X)$ by Proposition 6.4. We may assume that the infinite binary tree T properly embedded in \mathbb{R}^{n+1} is a full subcomplex of some triangulation of \mathbb{R}^{n+1} , which we fix from now on, and that R is the derived neighborhood of T . Since X is \hat{LC}_{n-1} , φ extends to a map $\partial R \cup K \rightarrow X$, also denoted φ , where K is the union of the dual cells to all but finitely many of the edges of T . Let B_0, B_1, \dots be the closures of the components of $R \setminus K$. Thus all but finitely many of B_i ’s are the dual cells of vertices of T , and the remaining ones are the derived neighborhoods of finite subtrees of T . Without loss of generality, B_0 is the derived neighborhood of a finite subtree $T_1 \subset T$, and B_1, B_2, \dots are the dual cells of vertices of T . In particular, all B_i ’s are PL balls.

Let $U_i = \{U_{i\alpha}\}$ be a sequence of covers of X such that each $U_{i\alpha}$ has diameter at most 2^{-i} and each $U_{i+1,\alpha}$ is contained in some $U_{i\beta}$. Without loss of generality, the diameter of X is at most 1, in which case we may assume that $U_0 = \{X\}$. Then T is the union of an increasing chain of finite (simplicial) subtrees $T_1 \subset T_2 \subset T_3 \subset \dots$ such that if B_j is the dual cell of a vertex of $T \setminus T_i$, then $\varphi(\partial B_j)$ is contained in some $U_{i+1,\beta}$. Let R_i be the derived neighborhood of T_i , and let B_{i1}, \dots, B_{ir_i} be the dual cells of the vertices of $T_{i+1} \setminus T_i$ (where $T_0 = \emptyset$). By reindexing and duplicating the elements of each U_i we may assume that each $\varphi(\partial B_{ij}) \subset U_{ij}$. Let $\varphi_0: S^n \rightarrow x \in X$ be the constant map. Assuming that $\varphi_i: (S^n, pt) \rightarrow (X, x)$ is defined, let us define φ_{i+1} by attaching each $\varphi|_{\partial B_{ij}}$ to φ_i along some path in U_{kl} from a point in $\varphi(\partial B_{ij})$ to a point in $\varphi_i(S^n)$, where U_{kl} contains U_{ij} and if $k < m < i$ then any U_{mp} containing U_{ij} is disjoint from $\varphi_i(S^n)$.

Since X is compact, $k = k(i) \rightarrow \infty$ as $i \rightarrow \infty$. Since each φ_{i+1} is $2^{-k(i)}$ -close to φ_i , the sequence $\varphi_1, \varphi_2, \dots$ uniformly converges to a continuous map $\varphi_\infty: (S^n, pt) \rightarrow (X, x)$. Let P_i be the nerve of the cover U_i (with duplicate elements omitted), let us fix simplicial bonding maps $P_{i+1} \rightarrow P_i$, arising due to the hypothesis that U_{i+1} refines U_i , and let $p_i^\infty: X \rightarrow P_i$ be the projection. Each $p_i^\infty \varphi_\infty: (S^n, pt) \rightarrow (P_i, p_i)$ is homotopic to $p_i^\infty \varphi_i$ by the “rectilinear” homotopy. Consecutive pairs of these homotopies yield homotopies $h_i^{(1)}$ between $p_i^\infty \varphi_{i+1}$ and $p_i^\infty \varphi_i$. A representative $\Phi: \text{Int} R \rightarrow P_{[0,\infty)}$ of $[\varphi]$ extending to a uniformly continuous map $\bar{\Phi}: R \rightarrow P_{[0,\infty]}$ agreeing with φ on ∂R can be obtained by combining some homeomorphisms $f_i: (S^n, pt) \rightarrow (\partial R_i, pt)$ with the “rectilinear” homotopies $h_i^{(2)}: (S^n, pt) \times I \rightarrow (P_{[i,i+1]}, p_{[i,i+1]})$ between the maps $p_i^\infty \varphi_{i+1}$ and $p_i^\infty \varphi_i$.

By construction, each $\varphi_i = (\psi_{11} + \dots + \psi_{1r_1}) + \dots + (\psi_{i1} + \dots + \psi_{ir_i})$, where each $\psi_{ij}: (S^n, pt) \rightarrow (X, x)$ is freely (i.e. unpointed) homotopic to $\varphi|_{\partial B_{ij}}$. Similarly, each $\varphi f_i = (\chi_{11} + \dots + \chi_{1r_1}) + \dots + (\chi_{i1} + \dots + \chi_{ir_i})$, where each $\chi_{ij}: (S^n, pt) \rightarrow (X, x)$ is freely homotopic to $\varphi|_{\partial B_{ij}}$. Since $\hat{\pi}_1(X)$ acts trivially on $\hat{\pi}_{n+1}(X)$, the forgetful map $\hat{\pi}_n(X) \rightarrow [S^n, X]^\Delta$ into the set of free homotopy classes is a bijection. So each ψ_{ij} is (pointed) homotopic to χ_{ij} by a homotopy h_{ij} , and consequently each

φ_i is homotopic to φf_i by a homotopy h_i . Moreover, if $\hat{\pi}_1(X) = 1$, the homotopies h_{ij} can be chosen to go through maps $(S^n, pt) \rightarrow (Y, 0) \rightarrow (X, x)$, where Y is the one-point compactification of $\mathbb{R}^n \sqcup [0, \infty)$, such that the image of \mathbb{R}^n is contained in U_{ij} . Hence each map $S^{n+1} \rightarrow P_i$ given by “rectilinear” null-homotopies of $p_i^\infty \psi_{ij}$ and $p_i^\infty \chi_{ij}$ along with the homotopy $p_i^\infty h_{ij}$ is null-homotopic (by a “rectilinear” homotopy). It follows that each $p_i^\infty h_{i+1}$ is homotopic to $p_i^\infty h_i$ by a “rectilinear” homotopy extending $h_i^{(1)}$ and $h_i^{(2)}$. \square

Theorem 6.5 and Theorem 6.1(g) have

Corollary 6.6. *Let X be a compactum with $\hat{\pi}_1(X) = 1$, and let $n \geq 1$. Then X is semi- LC_n iff X is LC_{n-1} and $\pi_n(X)$ is discrete.*

A 0-dimensional counterpart of Corollary 6.6 (without the assumption of simple connectedness) is obtained in §8 (see Corollary 8.8). There will follow a sketch of an alternative approach to proving an assertion, close to Corollary 6.6; the author hopes that this approach will help to shed light on the necessity of simple connectedness in the hypothesis of Corollary 6.6.

Let us now turn to homological versions of the above results.

Theorem 6.7 (Eda–Kawmura, Shchepin). *Let X be an LC_{n-1} compactum, $n \geq 1$. Then $\hat{H}_n(X) \xrightarrow{\hat{\tau}} H_n(X)$ is an epimorphism.*

The case $n = 1$ was established by Shchepin [MRS; Theorem 4.1] using Bestvina’s thesis and other non-trivial structure results in topology of compacta. The author learned from Shchepin about five years ago that he could also prove the general case of Theorem 6.7; this proof has never been written up. It should be noted that some key parts of the argument in [MRS] do not seem to generalize for arbitrary n , and the author does not know how these difficulties have been tackled by Shchepin. Eda and Kawamura proved the following corollary of Theorem 6.7: if X is $L\hat{C}_{n-1}$ and $n \geq 1$, $\hat{H}_n(X) \xrightarrow{\hat{\tau}} \check{H}_n(X)$ is an epimorphism [EK1].

The restriction $n \geq 1$ is necessary in Theorem 6.7 by Example 5.4.

Proof. Similarly to Proposition 6.4 one shows that if X is $L\hat{C}_{n-1}$ (or just $HL\hat{C}_{n-1}$), every element of $H_n(X)$, $n \geq 1$, can be represented by a uniformly continuous map $M \rightarrow X$ from an oriented n -pseudo-manifold that admits a proper ε -map $M \rightarrow T$ (i.e. a map whose point-inverses have diameters $< \varepsilon$) onto the infinite binary tree, with respect to some proper function $\varepsilon: T \rightarrow (0, 1]$. Since X is $L\hat{C}_{n-1}$ (or just $HL\hat{C}_{n-1}$), “each trunk” in this “cycle with trunks” can be partitioned into an infinite sum of small singular cycles. Then by Lemma 6.3(a), the $L\hat{C}_{n-1}$ condition allows to replace M with the sum of a singular compact pseudo-manifold $M_0 \rightarrow X$ and a finite number of spheroids with trunks $\partial R_i \rightarrow X$. The latter can be converted within their Steenrod homology classes (not within their Steenrod homotopy classes!) into genuine spheroids $S^n \rightarrow X$ by the proof of Theorem 6.5 (which is simplified as one does not have to keep track of basepoints now). \square

Theorem 6.8. *Fix some integer $n \geq 0$ and let X be a compactum.*

- (a) (Shchepin) $HL\hat{C}_n \Rightarrow HLC_n \Leftrightarrow HL\check{C}_n$.
- (b) (Jussila [Ju]; see also [Br; VI.10.6]) *If X is semi- HLC_n , then $H_n(X) \xrightarrow{\tilde{\tau}} \check{H}_n(X)$ is an isomorphism.*

- (c) If X is $HL\hat{C}_n$, then $\hat{H}_n(X) \xrightarrow{\hat{\tau}} H_n(X)$ is an isomorphism.
- (d) (Mardešić [Ma]; see also [Br; VI.12.6]) If X is semi- $HL\hat{C}_n$, then the composition $\hat{H}_n(X) \xrightarrow{\hat{\tau}\hat{\tau}} \check{H}_n(X)$ is an isomorphism.
- (e) If X is $HL\hat{C}_{n-1}$, then $\hat{H}_n(X) \xrightarrow{\hat{\tau}\hat{\tau}} \check{H}_n(X)$ is onto a dense subset.
- (f) X is semi- $HL\hat{C}_n$ iff X is $HL\hat{C}_{n-1}$ and $\hat{H}_n(X)$ is discrete.
- (g) X is semi- HLC_n iff X is HLC_{n-1} and $H_n(X)$ is discrete.
- (h) X is semi- $HL\check{C}_n$ iff X is $HL\check{C}_{n-1}$ and $\check{H}_n(X)$ is discrete.

Part (g) follows immediately from (a), (b) and (h). Part (c) follows from (a), (b) and (d) (compare [Br; V.12.15]), but we also give a direct proof below.

Part (a) was proved by Shchepin [MRS; Theorem 4.10] in the case $n = 1$. The author learned from Shchepin about five years ago that he could also prove the general case of (a); this proof has never been written up. While our proof of (a) is based on the same idea as the argument of [MRS], namely, the Hurewicz Construction (see the proof of Lemma 6.1(3)), our realization of this idea in higher dimensions depends on a more delicate technique of the geometric approach to cohomology, developed by Buoncrisiano, Rourke and Sanderson [BRS].

The implication $HL\hat{C}_\infty \Rightarrow HLC_\infty$ cannot be reversed for the cluster of copies of an arbitrary non-simply-connected acyclic 2-polyhedron [EKR].

Pseudo-comanifolds. We call a proper PL map $f: W \rightarrow P$ between polyhedra a (*co-connected*) *k-pseudo-comanifold* if there exists a triangulation of P such that the preimage $f^{-1}(\Delta^i)$ of every its i -simplex is a (pseudo-connected) $(i-k)$ -pseudo-manifold with boundary equal to the preimage of the boundary $f^{-1}(\partial\Delta^i)$. Such a triangulation of P will be called *transverse* to f . Co-manifolds originate in [BRS], where they are called “mock bundles”.

An embedded k -pseudo-comanifold $f: W \hookrightarrow P$ is *co-orientable* if $H_k(P, P \setminus W)$ contains no torsion. In this case $H^k(P, P \setminus W)$ is free abelian, and a choice of a set of its generators, representable by cocycles with disjoint supports, is called a *co-orientation* of f . (For reference, a k -pseudo-manifold M without boundary is orientable iff the compactly supported cohomology $H^k(M)$ is torsion-free. In this case the locally finite homology $H_k^{\text{lf}}(M)$ is free abelian, and a choice of a set of its generators, representable by cycles with disjoint supports, corresponds to an orientation of M .) If $f: W \rightarrow P$ is an arbitrary pseudo-comanifold, where P is compact, f is the projection of an embedded pseudo-comanifold $\bar{f}: W \rightarrow P \times \mathbb{R}^N$ for some N , and the *co-orientability* (*co-orientation*) of f can be well-defined as that of \bar{f} .

It is easy to see that $H^k(P; \mathbb{Z}/2)$ (respectively $H^k(P)$) is isomorphic to the group of (oriented) pseudo-cobordism classes of (oriented) k -pseudo-comanifolds in P , cf. [BRS].

Let $f: W \rightarrow P$ be an oriented k -pseudo-comanifold. Note that if P is an oriented i -pseudo-manifold with respect to a triangulation of P , transverse to f , then W will be an oriented $(i-k)$ -pseudo-manifold, cf. [BRS; II.1.2]. On the other hand, if $\varphi: M \rightarrow P$ is simplicial with respect to a triangulation of P , transverse to f , the pullback $\varphi^*(f): N \rightarrow M$ is an oriented k -pseudo-comanifold, cf. [BRS; bottom of p. 23]. Now if φ happens to be an oriented singular i -pseudo-manifold, we obtain an oriented singular $(i-k)$ -pseudo-manifold $f^!(\varphi): N \rightarrow W$. Using the techniques of

[BRS], it is easy to show that $f^!: H_i(P) \rightarrow H_{i-k}(W)$ is well-defined. Furthermore, $f_*f^![\varphi] = [f] \frown [\varphi]$, cf. [BRS; p. 29].

$$\begin{array}{ccc} N & \xrightarrow{f^!(\varphi)} & W \\ \varphi^*(f) \downarrow & & f \downarrow \\ M & \xrightarrow{\varphi} & P \end{array}$$

If $f: W \rightarrow P$ is a co-connected oriented 0-pseudo-comanifold, it is easy to see that $[f] = \pm[\text{id}_P] \in H^0(P)$. Let us call such an f an *elaboration* of the polyhedron P if specifically $[f] = [\text{id}_P] \in H^0(P)$. Then, in particular, $f_*f^! = \text{id}: H_i(P) \rightarrow H_i(P)$ for each i . Hence $H_i(P)$ can be identified with a direct summand of $H_i(W)$. In addition, using once again the co-connectedness of f , for each oriented singular i -pseudo-manifold $\varphi: M \rightarrow P$, simplicial with respect to a triangulation of P , transverse to f , the elaboration $\varphi^*(f): N \rightarrow M$ induces an isomorphism $H_i(N) \rightarrow H_i(M)$ due to the pseudo-connectedness of N .

Elaborated Ferry Construction. Assuming that X is $\text{HL}\hat{C}_n$, we replace the retraction $r_k^{(n+1)}: P_{[k,\infty)}^{(n+1)} \cup X \rightarrow X$ and its homotopy to the identity with values in $P_{[l,\infty]}$ keeping X fixed (see ‘‘Ferry’s Construction’’ in the beginning of this section) by the following data:

- (i) an elaboration $q^{(n+1)}: Q^{n+1} \rightarrow P_{[k,\infty)}^{(n+1)}$;
- (ii) a retraction $\bar{r}_k^{(n+1)}: Q^{n+1} \cup X \rightarrow X$;
- (iii) a pseudo-cobordism $W^{n+2} \rightarrow P_{[k,\infty)}^{(n+1)}$ between $q^{(n+1)}$ and $\text{id}_{P_{[k,\infty)}^{(n+1)}}$;
- (iv) a map $W^{n+2} \cup X \rightarrow P_{[l,\infty]}$ restricting to $\bar{r}_k^{(n+1)}$ on $Q^{n+1} \cup X$ and to the inclusion on $P_{[k,\infty)}^{(n+1)}$.

These data are constructed similarly to the Ferry Construction, item (iii) being granted by the definition of an elaboration.

If X is semi- $\text{HL}\hat{C}_n$, then $\bar{r}_k^{(n)}: Q^n \cup X \rightarrow X$ restricted to Q^n extends to a continuous map $Q^{n+1} \rightarrow X$.

Proof of Theorem 6.8. (c), (d), (e). Using the elaborated Ferry Construction, these parts can be now be proved similarly to the corresponding parts of Theorem 6.1. \square

Fractalized pseudo-manifolds. An inverse sequence $\dots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0$ of polyhedra and PL maps will be called *fractalizing* if each f_i is an elaboration, transverse to a triangulation T_i of P_i such that f_i is simplicial with respect to T_{i+1} and some subdivision of the barycentric subdivision of T_i . Such a sequence of triangulations (T_0, T_1, \dots) will also be called *fractalizing*. The limit F of such an inverse sequence is a *fractalized* P_0 and $p_0^\infty: F \rightarrow P_0$ is a *fractalization* of P_0 .

If Q is an n -pseudo-manifold and $f: M \rightarrow Q$ its fractalization, then by the above $f_*: H_n(M) \rightarrow H_n(Q)$ is an isomorphism, and in particular M has a well-defined fundamental class $[M]$ in Steenrod homology. If X is a compactum and $\varphi: M \rightarrow X$ is a map, we say that it *represents* $\varphi_*([M]) \in H_n(X)$.

Lemma 6.9. *Let X be a $HL\check{C}_n$ compactum, and let $\varphi: M \rightarrow X$ be a singular fractalized oriented n -pseudo-manifold. If $\tilde{\tau}([\varphi]) = 0 \in \check{H}_n(X)$, then φ bounds a singular fractalized oriented null-pseudo-bordism.*

The idea of the proof is based on the Hurewicz Construction (see the proof of Lemma 6.1(3)). The case $n = 1$ was obtained by Shchepin [MRS; Theorem 4.4]. This case is special in that 2-dimensional fractal pseudo-manifolds reduce to 2-dimensional fractal manifolds, which are all homeomorphic to each other, see [MRS].

Proof. We may assume that the assertion holds in dimensions $< n$. Suppose that M is the limit of a fractalizing inverse sequence $\dots \xrightarrow{f_1} Q_1 \xrightarrow{f_0} Q_0$ with Q_0 an oriented n -pseudo-manifold. Suppose that X is the inverse limit of polyhedra P_i and PL maps p_i . Without loss of generality, φ extends to a level-preserving map $\varphi_{[0,\infty]}: Q_{[0,\infty]} \rightarrow P_{[0,\infty]}$ (see §2). By an abuse of notation, let us redefine $Q_{[0,\infty]}$ by replacing each mapping cylinder $MC(f_i)$ with a pseudo-cobordism between the elaboration $f_i: Q_{i+1} \rightarrow Q_i$ and the identity of Q_i . Then $\varphi_{[0,\infty]}$ is a Steenrod cycle representing $[\varphi]$, thus by the hypothesis each φ_i bounds a singular oriented pseudo-manifold $\hat{\varphi}_i: \hat{Q}_i \rightarrow P_i$. Given an $\varepsilon > 0$, there exists a k such that there is a triangulation of the pair (\hat{Q}_k, Q_k) , extending a triangulation of Q_k in a fractalizing sequence of triangulations, and such that the image of every its simplex under $\hat{\varphi}_k$ is ε -close to X and has diameter at most ε . Note that for each simplex Δ of this triangulation of Q_k , the inverse sequence $\dots \rightarrow (f_k^{k+2})^{-1}(\Delta) \rightarrow (f_k^{k+1})^{-1}(\Delta) \rightarrow \Delta$ is fractalizing. Then similarly to Ferry's Construction (see the proof of Theorem 6.1), the inductive hypothesis implies that for each $\delta > 0$, the number ε can be chosen so that $\varphi_{[k+1,\infty]}$ extends to a level-preserving map $\psi_{[k+1,\infty]}: R_{[k+1,\infty]} \rightarrow P_{[k+1,\infty]}$, where the inverse sequence $\dots \xrightarrow{g_{k+1}} R_{k+1} \xrightarrow{g_k} R_k := Q_k \cup \hat{Q}_k^{(n)}$ is fractalizing, each $g_i^{-1}(Q_i) \subset R_{i+1}$ is identified with Q_{i+1} so that $g_i|_{Q_{i+1}}$ is identified with $f_i|_{Q_{i+1}}$, and $\psi_{[k+1,\infty]}$ is δ -close to the composition of the projection onto R_k and the restriction of $\hat{\varphi}_k$. (Here $R_{[k+1,\infty]}$ consists of pseudo-cobordisms rather than mapping cylinders.)

Let Δ be an $(n+1)$ -simplex of \hat{Q}_k , and let $S_i = (g_k^i)^{-1}(\partial\Delta) \subset R_i$ for each $i > k$. Let S_∞ be the limit of the fractalizing inverse sequence $\dots \rightarrow S_{k+1} \rightarrow S_k$. Then the restriction $\chi_{[k+1,\infty]}$ of $\psi_{[k+1,\infty]}$ to $S_{[k+1,\infty]}$ is a Steenrod cycle representing the image of the fundamental class of S_∞ in $H_n(X)$. Recall that the image of S_∞ in X has diameter at most δ . Since X is $HL\check{C}_n$, for each $\gamma > 0$ this δ can therefore be chosen so that $\chi_{[k+1,\infty]}$ be Čech trivial with support in the γ -neighborhood in X of the image of S_∞ .

Then each χ_i bounds a singular connected oriented pseudo-manifold $\hat{\chi}_i: \hat{S}_i \rightarrow P_i$ with image in the γ -neighborhood of $\chi_i(S_i)$. Given any specific value of i , we may extend the restriction $S_i \rightarrow \partial\Delta$ of g_k^i to a map $\hat{S}_i \rightarrow \Delta$, simplicial with respect to some subdivision of the barycentric subdivision of Δ . Repeating the same for each $(n+1)$ -simplex Δ of \hat{Q}_k , we obtain an extension of $g_k^i: R_i \rightarrow Q_k \cup \hat{Q}_k^{(n)}$ to an elaboration $\hat{g}_k^i: \hat{R}_i \rightarrow \hat{Q}_k$ such that the various $\hat{\chi}_i$ combine into a map $\hat{\psi}_i: \hat{R}_i \rightarrow P_j$. Since $\hat{\psi}_i$ and $\hat{\varphi}_k \hat{g}_k^i$ are γ -close, given an l , we may choose γ so that they extend to a map $\hat{\psi}_{[k,i]}: \hat{R}_{[k,i]} \rightarrow P_{[l,i]}$ of a pseudo-cobordism $\hat{g}_k^{[i,k]}: \hat{R}_{[k,i]} \rightarrow \hat{Q}_k$ between \hat{g}_k^i and the identity of \hat{Q}_k , which restricts to $\psi_{[k,i]}: R_{[k,i]} \rightarrow P_{[k,i]}$.

Now given any l' , we have a similar string of dependencies $l' \mapsto \gamma' \mapsto \delta' \mapsto \varepsilon' \mapsto$

k' , and the above procedure applies to $\hat{\chi}_i$ with $i = k'$ in place of $\hat{\psi}_k$. Proceeding in this fashion, we eventually obtain a fractalizing inverse sequence $(\cdots \rightarrow T_1 \rightarrow T_0) = (\cdots \rightarrow \hat{R}_i \rightarrow \hat{Q}_k)$ and a proper map $T_{[0,\infty)} \rightarrow P_{[l,\infty)}$, which by construction extends to a continuous map of the inverse limit $T_\infty \rightarrow X$ that yields the required fractalized oriented null-pseudo-bordism of φ . \square

Fractalized Ferry Construction. Assuming that X is a HLC_n compactum, one can apply Lemma 6.9 to construct:

- (i) a fractalizing inverse sequence $\cdots \xrightarrow{q_1} Q_1 \xrightarrow{q_0} Q_0 = P_{[k,\infty)}^{(n+1)}$ with inverse limit Q_∞ ;
- (ii) a retraction $\tilde{r}_k^{(n+1)}: Q_\infty \cup X \rightarrow X$;
- (iii) a ‘telescopic pseudo-cobordism’ $q_0^{[0,\infty]}: W_{[0,\infty]} \rightarrow P_{[k,\infty)}^{(n+1)}$, combining pseudo-cobordisms $q_i^{[i,i+1]}: W_{[i,i+1]} \rightarrow Q_i$ between q_i and id_{Q_i} (thus $W_{\mathbb{N} \cup \infty} = Q_{\mathbb{N} \cup \infty}$);
- (iv) a ‘telescopic homotopy’ $h_k^{(n+2)}: W_{[0,\infty]} \cup X \rightarrow P_{[l,\infty)}$ restricting to $\tilde{r}_k^{(n+1)}$ on $Q_\infty \cup X$ and to the inclusion on $P_{[k,\infty)}^{(n+1)}$.

Corollary 6.10. (Shchepin) *If X is a HLC_n compactum, every element of $H_n(X)$ is representable by a singular fractalized oriented n -pseudo-manifold.*

This follows from the fractalized Ferry Construction similarly to the proof of Lemma 6.2(2). The author learned from E. V. Shchepin about five years ago that he could prove a version of Corollary 6.10 using another (possibly different) notion of a fractalized pseudo-manifold, defined by a direct inductive construction.

Proof of Theorem 6.8, continued. (a). The implication $\text{HLC}_n \Rightarrow \text{HLC}_n$ is trivial. The converse implication follows from local versions of Corollary 6.10 and Lemma 6.9. The implication $\text{HLC}_n \Rightarrow \text{HLC}_n$ follows from (d), or alternatively the implication $\text{HLC}_n \Rightarrow \text{HLC}_n$ follows from (c). \square

(b). This is similar to the proof of Corollary 6.6, but easier.

Let us represent an element of $H_n(X)$ by a proper singular oriented $(n+1)$ -pseudo-manifold $\varphi_{[0,\infty)}: Q_{[0,\infty)} \rightarrow P_{[0,\infty)}$. (Here each $Q_{[i,i+1]}$ is an arbitrary oriented pseudo-bordism, rather than a mapping cylinder.) If the image of $[\varphi_{[0,\infty)}]$ under the composition $H_n(X) \xrightarrow{\tilde{\tau}} \check{H}_n(X) \xrightarrow{p_k^\infty} H_n(P_k)$ is trivial, we may assume that $\varphi_k: Q_k \rightarrow P_k$ bounds a singular oriented $(n+1)$ -pseudo-manifold $\hat{\varphi}_k: \hat{Q}_k \rightarrow P_k$. Without loss of generality, the Steenrod cycle $\varphi_{[k,\infty)} \cup \hat{\varphi}_k: Q_{[k,\infty)} \cup \hat{Q}_k \rightarrow P_{[0,\infty)}$ maps simplicially into $P_{[k,\infty)}^{(n+1)}$. Let us write $Q = Q_{[k,\infty)} \cup \hat{Q}_k$ and $\pi = \varphi_{[k,\infty)} \cup \hat{\varphi}_k: Q \rightarrow P_{[0,\infty)}$. Given an $\varepsilon > 0$, we may choose k so that the fractalized Ferry Construction applies, with $\tilde{r}_k^{(n)}$ sending $(q_0^\infty)^{-1}(\partial\Delta)$, for each $(n+1)$ -simplex Δ of $P_{[0,\infty)}$, into a subset of X of diameter $< \varepsilon$. Given an $(n+1)$ -simplex Δ of Q , let ψ_Δ denote the Steenrod cycle $\varphi|_\Delta \cup h_k^{(n+1)}|_{R_{[0,\infty)}}: \Delta \cup R_{[0,\infty)} \rightarrow P_{[0,\infty)}$, where $R_i = (q_0^i)^{-1}(\varphi(\partial\Delta))$ form a fractalizing inverse sequence $\cdots \rightarrow R_1 \rightarrow R_0$. Then $[\varphi] = [\bigsqcup \psi_\Delta]$, where Δ runs over all $(n+1)$ -simplices of Q , already on the level of locally finite simplicial chains of $P_{[0,\infty)}$.

Since X is compact, it can be covered by a finite collection of sets U_1, \dots, U_r of diameters at most 2ε such that each $\tilde{r}_k^{(n)}((q_0^\infty)^{-1}(\partial\Delta))$ is contained in U_i for some $i = i(\Delta)$. For each i , consider the Steenrod cycle $\psi_i := \bigsqcup_{i=i(\Delta)} \psi_\Delta$. We

have $[\varphi] = [\psi_1] + \dots + [\psi_r]$ already on the level of locally finite chains. On the other hand, each $[\psi_i]$ lies in the image of $H_n(U_i)$, and therefore is trivial by the hypothesis. This proves that the composition $H_n(X) \xrightarrow{\tilde{\tau}} \check{H}_n(X) \xrightarrow{p_k^\infty} H_n(P_k)$ is injective for some k , which implies the assertion. \square

(f), (h). These are proved similarly to the corresponding parts of Theorem 6.1. \square

We record one byproduct of the preceding discussion.

Theorem 6.10'. *If $n > 0$, Corollary 6.10 holds under the weaker hypothesis of HLC_{n-1} .*

Proof. This is similar to the proof of Theorem 6.5, but easier, since one does not need to worry about basepoints, and Proposition 6.4 can be replaced by the construction in the proof of Theorem 6.8(b). \square

Theorem 6.11. *Let X be a compactum, $n \geq 0$.*

(a) *The following are equivalent:*

- (i) *X is LC_n ;*
- (ii) *for any closed $Y, Z \subset X$ with $Y \subset \text{Int}Z$, the inclusion $Y \hookrightarrow Z$ factors through an LC_n compactum W ;*
- (iii) *for any closed $Y, Z \subset X$ with $Y \subset \text{Int}Z$, $\text{im}[\pi_i(Y) \rightarrow \pi_i(Z)]$ is countable for all $i \leq n$.*

(b) *The following are equivalent:*

- (i) *X is HLC_n ;*
- (ii) *for any for any closed $Y, Z \subset X$ with $Y \subset \text{Int}Z$, Y is contained in an LC_n compactum W such that the inclusion $Y \hookrightarrow Z$ factors through an inclusion of Y into a compactum \hat{W} such that $\hat{W} \setminus Y$ is a fractalization of $W \setminus Y$;*
- (iii) *for any closed $Y, Z \subset X$ with $Y \subset \text{Int}Z$, $\text{im}[H_i(Y) \rightarrow H_i(Z)]$ is finitely generated for all $i \leq n$.*

The implication (i) \Rightarrow (iii) in (b) is due to Borel and Moore [BoM], and its converse to Dydak [D6]. The earliest result of this kind is the cohomological analogue of (i) \Leftrightarrow (iii), which is due to Wilder (see [Br]).

Proof. (a). Assuming (i), let Y and Z be as in (ii), and suppose that $(X, Y) = \varprojlim (P_i, Q_i)$. By Ferry's Construction (see the proof of Theorem 6.1), $P_{[k, \infty)}^{(n+1)} \cup X$ retracts onto X for some k . Pick an $l > k$ such that this retraction sends $W := Q_{[l, \infty)}^{(n+1)} \cup Y$ into Z . Clearly, W is LC_n .

Assuming (ii), by parts (h) and (c) of Theorem 6.1, $\pi_i(W)$ is discrete for all $i \leq n$. Hence by Lemma 3.4(b), each $\pi_i(W)$ is countable, which implies (iii).

Assuming (iii), let $\dots \subset N_2 \subset N_1$ be a fundamental sequence of closed neighborhoods of an arbitrary given point $x \in X$. Then by Theorem 3.1(c), $\varprojlim \pi_k(N_i) = 0 = \varprojlim^1 \pi_k(N_i)$ for all k . If $G_i = \text{im}[\pi_k(N_{i+1}) \rightarrow \pi_k(N_i)]$, then we still have $\varprojlim G_i = 0 = \varprojlim^1 G_i$ for all k . Since G_i are countable for all $k \leq n$, by Lemmas 3.3 and 3.4(a), for each i there exists a $j > i$ such that $G_j \rightarrow G_i$ is trivial for all $k \leq n$. This implies (i). \square

(b). The proof of (iii) \Rightarrow (i) is similar to that in (a). The proof of (i) \Rightarrow (ii) is similar to that in (a), using the fractalized Ferry Construction.

Assuming (ii), similarly to the proof of (ii) \Rightarrow (iii) in (a), we get that $H_i(W)$ injects into a finitely generated abelian group. Hence it is finitely generated, and the assertion follows from the commutativity of the diagram

$$\begin{array}{ccccc} H_i(\hat{W}) & \xrightarrow{f_*} & H_i(W) & & \\ \text{\scriptsize incl}_* \uparrow & & f^! \downarrow & & \\ H_i(Y) & \xrightarrow{\text{\scriptsize incl}_*} & H_i(\hat{W}) & \xrightarrow{(\tilde{r}_k^{(n+1)})_*} & H_i(Z). \quad \square \end{array}$$

An inverse sequence of groups G_i is said to be *nearly Mittag-Leffler* if for each i there exists a $j > i$ such that for each $k > j$ the image of $G_j \rightarrow G_i$ is contained in the normal closure of the image of $G_k \rightarrow G_i$. If X is the limit of an inverse sequence of compact connected polyhedra P_i , the property that the inverse sequence $\pi_1(P_i)$ is nearly Mittag-Leffler clearly does not depend on the choice of the basepoint of X and is an invariant of the equivalence relation in Proposition 2.6(iii). Then we may call X *nearly Steenrod connected* if this property is satisfied. Thus the property of being nearly Steenrod connected is an (unpointed) shape invariant. It was introduced by McMillan [Mc] (under a different name); it is proved in [Mc] that a continuous image of a nearly Steenrod connected compactum is nearly Steenrod connected. Clearly (see Theorems 3.1(b), 4.1(iii) and Lemma 3.3),

$$\begin{array}{c} \text{Steenrod connected (i.e. } \pi_0(X) = 0) \\ \Downarrow \\ \text{nearly Steenrod connected} \\ \Downarrow \\ \text{homologically Steenrod connected (i.e. } H_0(X) = 0). \end{array}$$

Theorem 6.12. *Let X be the limit of an inverse sequence of compact connected polyhedra P_i . Let us consider the one-point compactification $P_{[0,\infty]}/X$ of $P_{[0,\infty]}$.*

- (a) (Shrikhande [Sh]) $P_{[0,\infty]}/X$ is LC_1 iff X is nearly Steenrod connected;
- (b) (Dydak [D6]) $P_{[0,\infty]}/X$ is HLC_n iff $H_i(X)$ is discrete for all $i < n$.

We note that by Theorem 6.1(b), $P_{[0,\infty]}/X$ is LC_n if and only if it is both LC_1 and HLC_n .

Proof of (b). The proof of Theorem 3.12 works to show that $H_k(X)$ is discrete for all $k < n$ iff for each i there exists a $j > i$ such that $H_k(P_{[j,\infty]}, X)$ maps trivially to $H_k(P_{[i,\infty]}, X)$ for each $k < n$. By the Map Excision Axiom 4.1(i), the latter is equivalent to $P_{[0,\infty]}/X$ being HLC_n at the compactifying point X/X . \square

The proof of (a) can be carried out along the lines of the proof of Theorem 3.12; we leave the details to the interested reader.

7. COVERING THEORY

Theorem 7.1. *Let X be a connected locally connected compactum and let $d \in \{1, 2, \dots, \infty\}$. The monodromy map yields a bijection between d -fold covering maps*

over X (up to a fiberwise homeomorphism) and representations of the topological group

- (a) $\hat{\pi}_1(X)$;
- (b) $\tilde{\pi}_1(X)$ (or $\pi_1(X)$)

into the discretely topologized symmetric group S_d (up to an inner automorphism of S_d). Connected covering spaces correspond to transitive representations.

If X is a semi-L \hat{C}_1 compactum, $\hat{\pi}_1(X)$ and $\tilde{\pi}_1(X)$ coincide and are discrete by parts (d), (f), (h) of Theorem 6.1, so the usual covering theory is a special case of each of the assertions (a) and (b) of Theorem 7.1.

Fox mentions Theorem 7.1(a) [F1; p. 2] with the following comment: “I did this in lectures at the University of Mexico in the summer of 1951. It has since been independently discovered by others, and appears for example in [Spanier’s textbook], p. 82”. Concerning those others, see in particular [AM].

Using Theorem 6.1(e), one can deduce Theorem 7.1(b) directly from 7.1(a):

Proof of (b). Corollary 2.5.3, Lemma 2.5.11 and Theorem 2.5.13 of [Sp] imply that fiberwise homeomorphism classes of connected d -fold covering maps $p: \tilde{X} \rightarrow X$ correspond bijectively to conjugacy classes of index d subgroups of the singular fundamental group $\hat{\pi}_1(X)$ that contain the preimage of some neighborhood of 1 in $\tilde{\pi}_1(X)$ under the composition $\hat{\pi}_1(X) \xrightarrow{\hat{\tau}} \pi_1(X) \xrightarrow{\tilde{\tau}} \tilde{\pi}_1(X)$. Since $\tilde{\pi}_1(X)$ is zero-dimensional, a base of neighborhoods of 1 is given by all clopen sets. Therefore the subgroups in question are precisely those that contain the kernel of $\tilde{\tau}\hat{\tau}$ composed with some continuous representation of $\tilde{\pi}_1(X)$ into a discrete group. On the other hand, by Theorem 6.1(e) the image of $\tilde{\tau}\hat{\tau}$ is dense in $\tilde{\pi}_1(X)$. Thus the fiberwise homeomorphism classes of connected d -fold covering spaces of X correspond bijectively to conjugacy classes of index d subgroups of $\tilde{\pi}_1(X)$ that contain the kernel of a continuous representation into a discrete group. If H is such an index d subgroup, the intersection K of all its conjugates contains the kernel into the discrete group, and so $\tilde{\pi}_1(X)/K$ is discrete. The action of $\tilde{\pi}_1(X)/K$ on the right cosets of its index d subgroup H/K yields a transitive representation $\tilde{\pi}_1(X)/K \rightarrow S_d$. This proves the transitive case of (b), which implies the general case using that a locally connected compactum has open connected components. \square

In order to generalize Theorem 7.1 to non-locally-connected spaces, Fox “corrected” the notion of a covering [F1], [F2]. A minor inaccuracy in Fox’s theory was in turn corrected in [Mo]. See also [MaM] and references there. We recall that a *covering map* is a map $p: \tilde{X} \rightarrow X$ such that there exists a cover $\{U_\alpha\}$ of X satisfying

- (i) each $p^{-1}(U_\alpha) = \bigsqcup_\lambda U_\alpha^\lambda$, where each $p|_{U_\alpha^\lambda}$ is a homeomorphism onto U_α .

In this case we shall say that the cover $\{U_\alpha^\lambda\}$ of \tilde{X} *lies over* $\{U_\alpha\}$. An *overlay structure* on p is a cover $\{U_\alpha^\lambda\}$ of \tilde{X} lying over some cover of X and additionally satisfying

- (ii) if $U_\alpha^\lambda \cap U_\beta^\mu$ and $U_\alpha^\lambda \cap U_\beta^\nu$ are both nonempty, then $\mu = \nu$.

Two overlay structures $\{U_\alpha^\lambda\}$ and $\{V_\alpha^\lambda\}$ on p are *equivalent* if there exists an overlay structure $\{W_\alpha^\lambda\}$ on p refining each of them (as covers of \tilde{X}). An *overlaying* is an equivalence class of overlay structures on a covering map. An *isomorphism*

of overlayings $[p: \tilde{X} \rightarrow X; \{U_\alpha^\lambda\}]$ and $[p': \tilde{X}' \rightarrow X; \{V_\beta^\lambda\}]$ is a homeomorphism $h: \tilde{X} \rightarrow \tilde{X}'$ that is fiberwise (i.e. $ph = p'$) and such that $\{h(U_\alpha^\lambda)\}$ and $\{V_\beta^\lambda\}$ are equivalent as overlay structures on p' .

Proposition 7.2. [F2], [Mo] *If X is locally connected, every covering over X admits a unique overlay structure.*

Moore's argument is helpful for comprehending the above definitions, so we reproduce it.

Proof [Mo]. If $\{V_\beta\}$ is a cover of X by connected sets, any two overlay structures on the given covering map p that lie over $\{V_\beta\}$ will clearly coincide up to a renumbering. To construct such a $\{V_\beta\}$, notice that every $x \in X$ has a connected neighborhood, since by the hypothesis the connected component of X containing x contains a neighborhood of x . Hence every cover $\{U_\alpha\}$ of X has a refinement $\{V_\beta\}$ where each V_β is connected. If $\{U_\alpha^\lambda\}$ is an overlay structure lying over $\{U_\alpha\}$, condition (ii) ensures that it has a refinement $\{V_\beta^\lambda\}$ lying over $\{V_\beta\}$. Thus any two overlay structures on a given covering map over X are equivalent.

On the other hand, suppose that $\{U_\alpha^\lambda\}$ lies over $\{U_\alpha\}$, and $\{V_\beta\}$ is a refinement of $\{U_\alpha\}$. If each V_β is connected, there exists a refinement $\{V_\beta^\lambda\}$ of $\{U_\alpha^\lambda\}$ lying over $\{V_\beta\}$. If additionally $V_\beta \cap V_\gamma \neq \emptyset$ implies $V_\beta \cup V_\gamma \subset U_\alpha$ for some α , then $\{V_\beta^\lambda\}$ will be an overlay structure. To construct such a $\{V_\beta\}$, let $\{W_\gamma\}$ be any cover of X by sets of diameters at most $\frac{\lambda}{2}$, where λ is a Lebesgue number for $\{U_\alpha\}$ (that is, every subset of X of diameter at most λ is contained in at least one U_α). Then $W_\beta \cap W_\gamma = \emptyset$ implies $W_\beta \cup W_\gamma \subset U_\alpha$ for some α . Finally $\{V_\beta\}$ is defined to be any refinement of $\{W_\gamma\}$ with connected elements. \square

There exists a covering over $\mathbb{N} \times I \cup [0, \infty) \times \partial I$ that admits no overlay structure [F2; Figs. 1, 2 and p. 78]. This is especially easy to see using Lemma 7.3(a), cf. [Mo; Example 1].

If X is the one-dimensional Hawaiian snail in Example 5.7, that is the mapping torus of the self-embedding of $\{0\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ defined by $x \mapsto \frac{x}{2}$, its universal covering admits uncountably many pairwise inequivalent overlay structures. However, they are all isomorphic to each other via covering transformations which are not self-isomorphisms of any overlay structure (compare [F1; Example 3]). Using this and Lemma 7.3(b), it is easy to construct a covering map over $S^1 \vee X$ (where the basepoint of X is not in the limit circle) that admits uncountably many pairwise non-isomorphic overlay structures [Mo; Example 3]. (Beware that [Mo; Example 2] is erroneous, as can be seen using either Lemma 7.3(a) or Corollary 7.5 or Theorem 7.6.)

The following lemma simplifies [F1; Extension Theorem 5.2], [Mo; Theorem 1.5].

Lemma 7.3. (a) *A covering map over a compactum admits an overlay structure iff it is induced from a covering map over a polyhedron.*

(b) *Overlayings $f^*(p)$, $g^*(q)$ over a compactum X induced from the covering maps p, q over polyhedra P, Q are isomorphic iff f and g factor up to homotopy as $f: X \xrightarrow{h} R \xrightarrow{f_1} P$ and $g: X \xrightarrow{h} R \xrightarrow{g_1} Q$, where R is a polyhedron and $f_1^*(p) = g_2^*(q)$.*

(a). Clearly, every covering map induced from an overlaying comes with the induced overlay structure. Conversely, suppose we are given a covering p over a compactum X with an overlay structure $\{U_\alpha^\beta\}$ lying over $\{U_\alpha\}$. If $U_\alpha^\lambda \cap U_{\beta_1}^{\mu_1} \cap \cdots \cap U_{\beta_r}^{\mu_r}$ and

$U_\alpha^\lambda \cap U_{\beta_1}^{\nu_1} \cap \cdots \cap U_{\beta_r}^{\nu_r}$ are both nonempty, each $\mu_i = \nu_i$. Therefore the obvious map q of the nerve of $\{U_\alpha^\lambda\}$ onto the nerve N of $\{U_\alpha\}$ is a covering map. Thus p is induced from the covering map q by the projection $X \rightarrow N$. \square

(b). Let $\{U_\alpha\}$ and $\{V_\beta\}$ be the covers of P and Q by the open stars of the vertices of some triangulations. Each $U_\alpha \cap U_\gamma$ and each $V_\beta \cap V_\delta$ are connected, so p and q admit overlay structures lying over $\{U_\alpha\}$ and $\{V_\beta\}$. Let $\{W_\gamma^\lambda\}$ be an overlay structure on the covering map $f^*(p) = g^*(q)$ that is a common refinement of the induced overlay structures. If $\{W_\gamma^\lambda\}$ lies over $\{W_\gamma\}$, then f and g factor up to homotopy through the nerve N of $\{W_\gamma\}$, and the maps $N \rightarrow P$ and $N \rightarrow Q$ induce the same covering map from p and q . \square

The following result represents a simplification of Fox's formulation of the classification of overlayings [F1].

Theorem 7.4. *Let X be a connected compactum and $d \in \{1, 2, \dots, \infty\}$. There exists a natural bijection between $[X, BS_d]$ and the set of isomorphism classes of d -fold overlayings over X .*

Note that when X is a polyhedron, this is a mere restatement of the polyhedral case of Theorem 7.1.

Proof. By Lemma 7.3(a), every overlaying over X is induced from a covering over a polyhedron P by a map $f: X \rightarrow P$. This yields a map $X \rightarrow P \rightarrow BS_d$, whose homotopy class is well-defined by Lemma 7.3(b). Conversely, if X is an inverse limit of polyhedra P_i , by Lemma 2.1(a) every map $f: X \rightarrow BS_d$ extends to $P_{[i, \infty]}$ for some i . Hence it factors up to homotopy through $X \rightarrow P_i$ and so gives rise to an overlaying. If f is homotopic to a $g: X \rightarrow BS_d$, which extends to $P_{[j, \infty]}$, by Lemma 2.1(b) the two extensions are homotopic over $P_{[k, \infty]}$ for some k and so determine the same overlaying. The assertion now follows from Proposition 2.3(a). \square

By Proposition 2.3(b), $[X, BS_d] = \varinjlim [P_i, BS_d]$ where the compactum X is the inverse limit of the polyhedra P_i . On the other hand, $\varinjlim [P_i, BS_d]$ can be identified with the orbit set of the action of S_d on $\varinjlim \text{Hom}(\pi_1(P_i), S_d)$ generated by inner automorphisms of S_d . It is well-known (and not hard to see) that Mittag-Leffler inverse sequences of groups reduce to their topological inverse limits [AS; §2]. This implies the following generalization of Theorem 7.1.

Corollary 7.5. [He] *Let X be a Steenrod connected compactum, and let $d \in \{1, 2, \dots, \infty\}$. The monodromy map yields a bijection between the isomorphism classes of d -fold overlayings over X and the conjugacy classes of representations of the topological group $\tilde{\pi}_1(X)$ (or $\pi_1(X)$) into the discretely topologized symmetric group S_d .*

Corollary 7.5 fails for arbitrary compacta: there is a free action of $\mathbb{Z}/3$ on the 2-adic solenoid Σ_2 with orbit space again homeomorphic to Σ_2 ; however $\tilde{\pi}_1(\Sigma_2) = 0$.

Note that the addendum on transitive representations, included in Theorem 7.1, is dropped in Corollary 7.5: in fact, it is false in this generality, by considering the one-dimensional Hawaiian snail (see [F1; Example 3], compare [F2]). It can be shown to hold, however, if “connected” is understood in the sense of uniform spaces rather than topological spaces:

Theorem 7.6. *Let X be a compactum. There is a natural bijection between the set of equivalence classes of overlay structures on a covering map $p: \tilde{X} \rightarrow X$ and the set of uniformities on \tilde{X} agreeing with p .*

We say that a uniformity (=uniform structure) on \tilde{X} agrees with a covering map $p: \tilde{X} \rightarrow X$, where X is a compactum, if every $x \in X$ has a closed neighborhood U such that

(iii) $p^{-1}(U) = \bigsqcup_{\lambda} U^{\lambda}$ as uniform spaces, where each $p|_{U^{\lambda}}$ is a homeomorphism.

Thus Fox's conditions (i) and (ii) are effectively replaced by the single condition (iii). An equivalent way of formulating (iii) is:

(iii') $p^{-1}(U)$ is uniformly homeomorphic to $U \times F$, where F is discrete, by a homeomorphism h such that $ph^{-1}: U \times F \rightarrow U$ is the projection.

We emphasize that F is assumed to be discrete as a uniform space, which is a stronger condition than the discreteness of its underlying topology. For instance, $\{\log 1, \log 2, \dots\} \subset \mathbb{R}$ is discrete as a topological subspace of \mathbb{R} , but not discrete as a uniform subspace of \mathbb{R} since $(\log(n+1) - \log n) \rightarrow 0$ as $n \rightarrow \infty$. Our reference for uniform spaces is Isbell's book [Is].

The above definition is equivalent to I. M. James' notion of covering in the uniform category (see [BDLM] where James' definition is slightly corrected). Theorem 7.6 may be viewed as an improvement of a result of [Mo], but the author was led to it through the analysis of Example 5.8.

Proof. Suppose that X is the inverse limit of compact polyhedra P_i . We may assume that they are triangulated so that each bonding map is simplicial as a map $P_{i+1} \rightarrow P_i$ into some subdivision of the barycentric subdivision of P_i . By Lemma 7.3(a) (see also the proof of Theorem 7.4) we may assume that the given overlaying p is induced from a covering map q_k over some P_k . For each $j > k$ let q_j be the covering map over P_j induced from q_k , and let E_j be its total space. Then \tilde{X} is the inverse limit of the uniformly continuous maps $E_{j+1} \rightarrow E_j$, where each E_j is endowed with the standard metric corresponding to a triangulation of E_j making q_j simplicial. This defines a uniformity on \tilde{X} agreeing with p , and by Lemma 7.3(a) (see also the proof of Theorem 7.4) it does not depend on the choice of q_k .

Given a uniformity on \tilde{X} agreeing with p , let $\{U_{\alpha}\}$ be a cover of X with each U_{α} satisfying (iii'). Without loss of generality it is a finite cover. Fix some metric on X , and let d be the diameter of X . Consider the metric on F where the distance between any two points is $2d$. Let us represent the uniformity on \tilde{X} by some metric. Let $\delta > 0$ be such that δ -close points of \tilde{X} remain d -close under each $h_{\alpha}: p^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times F$, and let $\varepsilon > 0$ be such that ε -close points of each $U_{\alpha} \times F$ remain δ -close under h_{α}^{-1} . Pick a refinement $\{V_{\beta}\}$ of $\{U_{\alpha}\}$ such that each $\{V_{\beta}\}$ has diameter at most ε . If V_{β} is contained in both U_{α} and U_{γ} , the restrictions $V_{\beta} \times F \xrightarrow{h_{\alpha}^{-1}} p^{-1}(V_{\beta}) \xrightarrow{h_{\gamma}} V_{\beta} \times F$ have to be of the form $\text{id}_{V_{\beta}} \times \sigma$ for some permutation $\sigma \in S_F$. Thus if $V_{\beta}^{\lambda} = h_{\alpha}(V_{\beta} \times \{\lambda\})$, the cover $\{V_{\beta}^{\lambda}\}$ of \tilde{X} is well-defined. If $\{V_{\beta}\}$ is chosen so that additionally $V_{\beta} \cap V_{\gamma} \neq \emptyset$ implies $V_{\beta} \cup V_{\gamma} \subset U_{\alpha}$ for some α (see the proof of Proposition 7.2), then $\{V_{\beta}^{\lambda}\}$ is an overlay structure. It is clear that any two overlay structures defined in this way are equivalent. \square

Corollary 7.7. *A fiberwise homeomorphism of overlayings is an isomorphism iff it is a uniform homeomorphism of the corresponding uniformities.*

Corollary 7.8. [F2], [Mo] *Every finite-sheeted covering map over a compactum X admits a unique overlay structure.*

The remainder of this section is devoted to an outline of how overlaying spaces of compacta can be included in the Steenrod homotopy category. These constructions will be used sporadically in the next section (most importantly in the proof of Theorem 8.9), and in turn assume the reader's familiarity with metrizable uniform spaces (see [Is]).

Uniform compactified mapping telescope. If X is a metrizable uniform space, the *cone* $CX := X \times I / X \times \{0\}$ is given the quotient uniformity, which is metrizable (see [Vi]) — in contrast to the quotient topology. The *mapping cylinder* $MC(f)$ of a uniformly continuous map $f: X \rightarrow Y$ between metrizable uniform spaces is the image of $\Gamma_f \times I \subset X \times Y \times I$ in the metrizable uniform space $CX \times Y$. If X is complete, then so is CX , and if additionally Y is complete, so is $MC(f)$.

Next, if A is a closed subset of a metrizable uniform space X , the quotient uniform space X/A is metrizable (see [Vi]). It follows, by standard techniques of uniform absolute retracts (see [Is], where they are called “injective spaces”) that every bounded metric on A , inducing its subspace uniformity, extends to a bounded metric on X , inducing its original uniformity. (The similar extension lemma for pseudo-metrics is known [Is; III.16].)

The *gluing* $X \cup_h Y$ of metrizable uniform spaces along a uniform homeomorphism h between a closed subset of X and a closed subset of Y is given the quotient uniformity $(X \sqcup Y)/h$, which is metrizable (see [BH; I.5.24]), and is well-defined by the above-mentioned extension lemma. If X and Y are complete, then so is $X \cup_h Y$.

Consider an inverse sequence $X = (\dots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0)$ of uniformly continuous maps between metrizable uniform spaces. Let $X_{[0,\infty]}$ be the inverse limit of the finite mapping telescopes $X_{[0,n]} = MC(f_0) \cup_{X_1} MC(f_1) \cup_{X_2} \dots \cup_{X_{n-1}} MC(f_{n-1})$ and the obvious retractions $X_{[0,n+1]} \rightarrow X_{[0,n]}$. Given a $J \subset [0, \infty]$, by X_J we denote the preimage of J under the obvious projection $X_{[0,\infty]} \rightarrow [0, \infty]$. According to the above, if each X_i is complete, then so is $X_{[0,\infty]}$. In that case it is the completion of the infinite mapping telescope $X_{[0,\infty)}$.

Convergent and Cauchy inverse sequences. In the notation of the preceding paragraph, let us call the inverse sequence X *convergent* if every uniform neighborhood of $\lim X$ in $X_{\mathbb{N} \cup \infty}$ (or equivalently in $X_{[0,\infty]}$) contains all but finitely many of X_i 's. We say that X is *Cauchy* if for every $\varepsilon > 0$ there exists a k such that for every $j > k$, the ε -neighborhood of X_j in $X_{\mathbb{N}}$ (or equivalently in $X_{[0,\infty)}$) contains X_k . Clearly, these notions depend only on the underlying uniform structures. Here is an example of a Cauchy inverse sequence that is divergent: $\dots \subset (0, \frac{1}{4}] \subset (0, \frac{1}{2}] \subset (0, 1]$. The inverse sequence $\dots \subset [2, \infty) \subset [1, \infty) \subset [0, \infty)$ fails to be Cauchy.²⁰

The close analogy with the definition of a convergent/Cauchy sequence of points in a metrizable uniform space can in fact be formalized. If M is a metric space, the space 2^M of all nonempty closed subsets of M is endowed with the Hausdorff metric

²⁰The reader who is acquainted with Mardešić's “resolutions” can easily check that a convergent inverse sequence of uniformly continuous maps between metrizable complete uniform spaces is a Mardešić resolution.

$d_H(A_1, A_2) = \min(\max(d_1, d_2), 1)$, where $d_i = \sup\{d(a, A_i) \mid a \in A_j, j \neq i\}$. If Y is a metrizable uniform space, the induced uniform structure of 2^Y is well-defined, and if Y is complete or compact, so is 2^Y [Is; II.48, II.49]. We obtain that the inverse sequence X is convergent (Cauchy) iff the sequence of the closed subsets X_i of $X_{\mathbb{N} \cup \infty}$ is convergent (Cauchy) as a sequence of points in $2^{X_{\mathbb{N} \cup \infty}}$, or equivalently in $2^{X_{[0, \infty]}}$. This implies parts (a) and (b) of the following lemma.

Lemma 7.9. *Let $X = (\dots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0)$ be an inverse sequence of uniformly continuous maps between metrizable uniform spaces.*

- (a) *If each X_i is compact, X converges.*
- (b) *If X converges, it is Cauchy; the converse holds when each X_i is complete.*
- (c) *X is convergent if and only if for each i , every uniform neighborhood of $f_i^\infty(\lim X)$ in X_i contains all but finitely many of $f_i^j(X_j)$'s.*
- (d) *X is Cauchy if and only if for each i and every $\varepsilon > 0$ there exists a k such that for every $j > k$, the ε -neighborhood of $f_i^j(X_j)$ in X_i contains $f_i^k(X_k)$.*
- (e) *If each X_i is uniformly discrete, X converges if and only if it satisfies the Mittag-Leffler condition.*
- (f) *If X converges and each X_i is non-empty (resp. uniformly connected), then $\lim X$ is non-empty (resp. uniformly connected).*
- (g) *If $Y = (\dots \xrightarrow{f_1} Y_1 \xrightarrow{f_0} Y_0)$ is another inverse sequence of uniformly continuous maps between metric spaces, $f_i: X_i \rightarrow Y_i$ are surjections commuting with the bonding maps and X is convergent (Cauchy), then Y is convergent (resp. Cauchy).*

Proof. (a). We have already deduced (a) from general facts, but it may be of interest to have an elementary proof. If X diverges, there exists an open neighborhood U of X in $X_{\mathbb{N} \cup \infty}$ whose complement has a nonempty intersection C_i with $X_{\{i, i+1, \dots, \infty\}}$ for each i . By Cantor's theorem, the compacta C_i have a non-empty intersection. It is contained in $\bigcap X_{\{i, i+1, \dots, \infty\}} = X$ and at the same time in the complement of U , which is a contradiction. \square

(c). A cover of $\lim X_{\{0, \dots, i\}}$ is uniform iff it can be refined by the preimage of a uniform cover of some $X_{\{0, \dots, i\}}$ (see [Is]). Hence a neighborhood of X_∞ in $X_{\mathbb{N} \cup \{\infty\}}$ is uniform iff it contains the preimage V of some uniform neighborhood U of $f_i^\infty(X_\infty)$ in $X_{\{0, \dots, i\}}$ for some i . But this U contains $f_i^j(X_j)$ (which are subsets of $X_i \subset X_{\{0, \dots, i\}}$) for almost all $j > i$ iff V contains X_j for almost all $j > i$. \square

(d)-(g). Part (d) is proved similarly to (c). Parts (e) and (f) follow from (c). Part (g) follows from (c) and (d). \square

Corollary 7.10 (Bourbaki's Mittag-Leffler Theorem). *Let L be the limit of an inverse sequence $X = (\dots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0)$ of uniformly continuous maps between complete metrizable uniform spaces. If each $f_i(X_{i+1})$ is dense in X_i , then $f_0^\infty(L)$ is dense in X_0 .*

Proof. The hypothesis implies that each $f_i^j(X_j)$ is dense in X_i . Hence by Lemma 7.9(d), X is Cauchy. Then by 7.9(b), X converges. If U is a uniform neighborhood of $f_0^\infty(L)$, by 7.9(c) it contains $f_0^j(X_j)$ for some j . If U is closed, this implies $U = X_0$. Since U is arbitrary, $f_0^\infty(L)$ is dense in X_0 . \square

Corollary 7.10 is known to generalize for inverse spectra rather than inverse sequences (see Bourbaki’s “General Topology”) and, in another direction, for continuous, rather than uniformly continuous bonding maps [Ru]. By a slight modification of an argument from [Ru] we also obtain

Corollary 7.11 (Baire’s Category Theorem). *The intersection of countably many dense open sets in a complete metrizable uniform space is dense.*

Proof. Without loss of generality, the dense open sets U_i are ordered by inclusion: $U_1 \supset U_2 \supset \dots$. If d is a metric on $U_0 := X$, let $d_0 = d$, and define a metric on U_{i+1} by $d_{i+1}(x, y) = d_i(x, y) + |\frac{1}{d_i(x, U_i \setminus U_{i+1})} - \frac{1}{d_i(y, U_i \setminus U_{i+1})}|$. Then d_{i+1} is a complete metric on U_{i+1} such that $\text{id}: (U_{i+1}, d_{i+1}) \rightarrow (U_{i+1}, d_i)$ is a uniformly continuous homeomorphism.

The assertion follows from the Mittag-Leffler theorem applied to the inverse sequence $\dots \rightarrow (U_1, d_1) \rightarrow (U_0, d_0)$. \square

Steenrod homotopy category extended. We call a uniform space X *residually finite-dimensional* if every uniform cover of X has a refinement of finite multiplicity (that is, with a finite-dimensional nerve); such spaces are called “finitistic” in [SSG]. A metrizable uniform structure is defined on a finite-dimensional uniform polyhedron in [Is]; it is complete. Residually finite-dimensional metrizable complete uniform spaces are precisely the limits of convergent inverse sequences of finite-dimensional uniform polyhedra and uniformly continuous bonding maps [Is; V.33]. In particular, these include overlaying spaces over compacta.

Let X and Y be the limits of convergent inverse sequences $\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0$ and $\dots \xrightarrow{q_1} Q_1 \xrightarrow{q_0} Q_0$ of finite-dimensional uniform polyhedra and uniformly continuous bonding maps. Without loss of generality, $P_0 = pt$ and $Q_0 = pt$. Then similarly to the proof of Lemma 2.1, every uniformly continuous map $f: X \rightarrow Y$ extends to a uniformly continuous map $f_{[0, \infty]}: P_{[0, \infty]} \rightarrow Q_{[0, \infty]}$ sending $P_{[0, \infty]}$ into $Q_{[0, \infty]}$. Moreover, every two such extensions are homotopic by a uniformly continuous (as a map $P_{[0, \infty]} \times I \rightarrow Q_{[0, \infty]}$) homotopy $\text{rel } X$, sending $P_{[0, \infty]} \times I$ into $Q_{[0, \infty]}$. We recall that we agreed to call a map $F: P_{[0, \infty]} \rightarrow Q_{[0, \infty]}$ *semi-proper* if for each k there exists an l such that $f^{-1}(Q_{[0, k]}) \subset P_{[0, l]}$.

We define a *Steenrod homotopy class* $X \rightsquigarrow Y$ to be the semi-proper homotopy class of a semi-proper map $f: P_{[0, \infty]} \rightarrow Q_{[0, \infty]}$. By the virtue of the semi-proper homotopy equivalences $(\text{id}_X)_{[0, \infty]}$ and $(\text{id}_Y)_{[0, \infty]}$ this definition does not depend on the choice of the inverse sequences P and Q . Steenrod homotopy groups of a residually finite-dimensional metrizable complete uniform space are now defined similarly to §3 (a special case was considered in the Introduction). Also, similarly to §4 one can define Steenrod homology and Pontryagin cohomology of such spaces, but we shall not need them for the purposes of this paper.

Universal generalized overlaying. Let X be a Steenrod connected compactum (the general case of a connected compactum will be a recurring theme throughout the next section). If X is the limit of an inverse sequence of compact connected polyhedra P_i , let $\tilde{X} = \varprojlim (\dots \rightarrow \tilde{P}_1 \rightarrow \tilde{P}_0)$, where \tilde{P}_i are their universal covering spaces. They come endowed with uniform structures (inherited from P_i ’s via the covering maps). Since P_i converge, so do \tilde{P}_i . It follows that the Steenrod homotopy type of the residually finite-dimensional metrizable complete uniform space \tilde{X} does not depend on the choice of P_i ’s. With some care, it can be shown that even the

fiberwise uniform homeomorphism class of \tilde{X} does not depend on this choice [LaB] (an alternative proof should appear as a byproduct in a subsequent paper by the present author). By definition, $p: \tilde{X} \rightarrow X$ is a Steenrod fibration; it is also a Serre fibration (see condition (ii) in the discussion preceding the statement of Theorem 3.15). Each fiber $p^{-1}(pt)$ is uniformly 0-dimensional, i.e. is an inverse limit of (countable) discrete uniform spaces. Obviously, $\pi_0(\tilde{X}) = pt$ and $\pi_1(\tilde{X}) = 1$. If $\tilde{\pi}_1(X)$ happens to be discrete, then p is an overlaying, which may be called the *universal overlaying* of X .

Example 7.12 (the Nottingham compactum). If R is a commutative associative ring with a unit, the set of formal power series $f(x) = x + a_2x^2 + a_3x^3 + \dots$ with coefficients $a_i \in R$ forms a group under composition: $fg(x) = f(g(x))$. This group $N(R)$ acts effectively and R -linearly on the formal power series ring $R[[x]]$. In the case $R = \mathbb{Z}_p$ (the p -adic integers) $N(R)$ is known to algebraists as the *Nottingham group*, and in the case $R = \mathbb{Z}$ it is related to the algebra of stable cohomology operations in complex cobordism [BSh], [Bu]. If R is a topological ring, by endowing $N(R)$ with the topology of the product $\prod_{i=2}^{\infty} R$ we make it into a topological group. Let $N_n(R)$ be the subgroup of $N(R)$ consisting of all power series with $a_2 = \dots = a_n = 0$, and let $N^n(R)$ denote the quotient $N(R)/N_n(R)$. Clearly, $N(R)$ is the inverse limit of the truncating maps $\dots \rightarrow N^2(R) \rightarrow N^1(R) = 1$.

Now let us consider the topological quotient groups $X = N(\mathbb{R})/N(\mathbb{Z})$ and $P_n = N^n(\mathbb{R})/N^n(\mathbb{Z})$. Since the bonding maps in $\dots \rightarrow N^2(\mathbb{Z}) \rightarrow N^1(\mathbb{Z})$ are surjective, X is the inverse limit of $\dots \rightarrow P_2 \rightarrow P_1$.

The universal covers of P_i are given by $N^i(\mathbb{R})$, hence $N(\mathbb{R})$ is a universal generalized overlay of X . Since $p: N(\mathbb{R}) \rightarrow X$ is a fibration and $N(\mathbb{R})$ is obviously contractible, it follows that $\hat{\pi}_1(X) \simeq N(\mathbb{Z})$ (as topological groups) and $\hat{\pi}_i(X) = 0$ for $i > 1$. (This was originally observed in [BB].) Since $p: N(\mathbb{R}) \rightarrow X$ is also a Steenrod fibration, we similarly conclude (compare Theorem 3.15(b)) that the Steenrod homotopy groups of X are the same; moreover, $\hat{\tau}: \hat{\pi}_i(X) \rightarrow \pi_i(X)$ is an isomorphism for all i .

From the algebraic viewpoint, this unexpected coincidence of singular and Steenrod homotopy is only made possible by the fact that the group $N(\mathbb{Z}) = \pi_1(X)$ is residually nilpotent.²¹

Indeed, let us consider the following embedding of the Hawaiian earring E (see §5) into X . Since each truncating map $p_n: P_{n+1} \rightarrow P_n$ is a bundle with fiber S^1 , we can embed the $(n+1)$ -fold wedge of circles Q_{n+1} into P_{n+1} by taking the union of some cross-section of p_n over the n -fold wedge Q_n and the fiber of p_n over the basepoint of that wedge. This yields an embedding of the inverse limit E of Q_n 's into X .

Recall from Example 5.6, that the image of $\hat{\tau}: \hat{\pi}_1(E) \rightarrow \pi_1(E)$ does not contain, for instance, the class of the following Steenrod loop ℓ :

$$(\dots, [a_1, a_2][a_1, a_3][a_1, a_4], [a_1, a_2][a_1, a_3], [a_1, a_2]) \in \varprojlim (\dots \rightarrow F_4 \rightarrow F_3 \rightarrow F_2).$$

This class is not representable by a map $S^1 \rightarrow E$ simply because such a map cannot wind infinitely many times around the first circle of E , whereas the letter a_1 is read off by the Pontryagin–Thom construction precisely on each pass of the first circle in the positive direction.

²¹and by the asphericity of X — see Example 8.6 and the subsequent remark.

What is the class of ℓ in $\pi_1(X)$? It is not hard to check that (in the group $N(R)$ with arbitrary R) if $\alpha = x + ax^n + \dots$ and $\beta = x + bx^m + \dots$, then $\alpha^{-1}\beta^{-1}\alpha\beta = x + ab(m-n)x^{m+n-1} + \dots$, where the dots represent terms of higher degrees, cf. [Jo].²² Thus, due to the relations in $N(\mathbb{Z})$, the commutators $[a_1, a_i]$ find themselves in increasingly deep subgroups $N_i(\mathbb{Z})$ of this group and hence are represented by increasingly short (exponentially shortening) loops. So their product is representable by a loop, in agreement with our expectations.

Remark. Let us note incidentally that the universal generalized overlay \tilde{E} of the Hawaiian Earring is not path-connected. Indeed, let F be the fiber of the Steenrod fibration $\tilde{E} \rightarrow E$. In the notation of the preceding example, the image of $[\ell]$ in $\pi_0(F)$ (see Theorem 3.15(b)) is representable by a map $t: S^0 \rightarrow F$ due to the 0-dimensionality of F . Now the image of $[t]$ in $\hat{\pi}_0(\tilde{E})$ is nontrivial by virtue of our choice of ℓ and using that \tilde{E} is Steenrod simply-connected.

8. ZERO-DIMENSIONAL HOMOTOPY

In the zero-dimensional case, Theorem 3.12 admits the following modification:

Proposition 8.1. (Krasinkiewicz [K2; 1.1]) *Let X be the limit of an inverse sequence of compact polyhedra $\dots \rightarrow P_2 \rightarrow P_1$. Then X is Steenrod connected iff for each k there exists a $j > k$ such that for every $i > j$, every path $(I, \partial I) \rightarrow (P_{[j, \infty]}, X)$ is homotopic rel ∂I and with values in $P_{[k, \infty]}$ to a path in $P_{[i, \infty]}$.*

Krasinkiewicz's original proof was geometric.

Proof. By Theorem 3.1 and Lemma 3.3, X is Steenrod connected iff $\pi_1(P_i)$ satisfy the Mittag-Leffler condition. By Lemma 3.11, this is the case iff $\hat{\pi}_1(P_{[i, \infty]}, X)$ satisfy the Mittag-Leffler condition. A homotopy h of a path can be converted to one that keeps the second endpoint fixed by using the restriction of h to that endpoint. \square

Theorem 8.2. (McMillan [Mc]; Krasinkiewicz [DS1; 7.2.4], [K2]) *A continuous image of a Steenrod connected compactum is Steenrod connected.*

At the same time, a continuous surjection between compacta need not induce an epimorphism on π_0 , by considering the projection $\Sigma_p \sqcup \Sigma_p \rightarrow \Sigma_p \vee \Sigma_p$ (see Examples 5.3 and 5.5).

The proof below simplifies McMillan's approach by using Proposition 8.1. It is also much easier than either of the two proofs by Krasinkiewicz. Yet another proof, involving non-metrizable compacta, is given in the third paper of the series [KO].

Proof. Let $f: X \rightarrow Y$ be the given surjection. By the proof of Theorem 3.15(a), we may assume that $X = \varprojlim P_i$, $Y = \varprojlim Q_i$ and $f = \varprojlim f_i$ for some compact polyhedra P_i and Q_i and PL maps $f_i: P_i \rightarrow Q_i$ commuting with the bonding maps. Pick a point $v \in Y$ and let v_k be its image in Q_k . Choose polyhedral neighborhoods $N_k(v)$ of v_k in Q_k so that each $N_k(v)$ contains the image of $N_{k+1}(v)$ and so that $\varprojlim N_i(v) = \{v\}$. Let $M_k(v) = f_k^{-1}(N_k(v))$, then $\varprojlim M_i(v) = f^{-1}(v)$; in particular, we may assume that each $M_i(v)$ is nonempty for sufficiently large $i = i(v)$.

²²This equality immediately implies that each $N^n(R)$ is nilpotent.

By the Ferry Construction for $\hat{\text{LC}}_{-1}$ compacta (see the beginning of §6), for each k there exist a j and an n such that a path $\ell: (I, \partial I) \rightarrow (Q_{[j,\infty]}, Y)$ is homotopic rel ∂ with values in $Q_{[k,\infty]}$ to an ℓ' such that $y_i := \ell'(\frac{i}{n}) \in Y$ for each $i = 0, \dots, n$, and $q_k^\infty(y_i) \in N_k(y_{i+1})$ for each $i < n$. Then there exists an $h > k$ such that $q_k^h(N_h(y_i)) \subset N_k(y_{i+1})$ for each $i < n$. Pick some $x_i \in M_h(y_i)$ for each $i = 0, \dots, n$. Note that $p_k^h(x_i) \in M_k(y_{i+1})$. By the Ferry Construction for $\hat{\text{LC}}_{-1}$ compacta, given an l , we may choose k so that x_i is joined to some $x_i^- \in f^{-1}(y_i)$ by a path in $M_{[l,\infty]}(y_i)$ and to some $x_i^+ \in f^{-1}(y_{i+1})$ by a path in $M_{[l,\infty]}(y_{i+1})$. Let $\ell_i: (I, \partial I) \rightarrow (P_{[0,\infty]}, X)$ be the product of these two paths, connecting x_i^- with x_i^+ . Since $Q_{[0,\infty]}$ is $\hat{\text{LC}}_\infty$, given an m , we may choose l so that $f_{[0,\infty]}(\ell_i)$ and $\ell'|_{[\frac{i}{n}, \frac{i+1}{n}]}$ are homotopic rel ∂ , with values in $Q_{[m,\infty]}$. On the other hand, by Proposition 8.1, we may choose $l = l(m)$ so that for each t , each ℓ_i is homotopic rel ∂ with values in $P_{[m,\infty]}$ to a path in $P_{[t,\infty]}$. \square

Theorem 8.3. (Krasinkiewicz [K2]) *A connected union of two Steenrod connected compacta is Steenrod connected.*

This assertion is a special case of Corollary 3.14; nevertheless, since it will be essentially used below, we include a slightly different, more direct proof.

Proof. We may represent X as $\varprojlim (\cdots \rightarrow P_2 \rightarrow P_1)$, where each P_i is a union of two compact polyhedra Q_i and R_i such that $Y = \varprojlim Q_i$ and $Z = \varprojlim R_i$. By the Ferry Construction for $\hat{\text{LC}}_{-1}$ compacta (see the beginning of §6), for each k there exists a j such that every path $l: (I, \partial I) \rightarrow (P_{[j,\infty]}, X)$ is homotopic rel ∂I and with values in $P_{[k,\infty]}$ to a product of paths ℓ_i alternating between $(Q_{[k,\infty]}, Y)$ and $(R_{[k,\infty]}, Z)$. Given an l , by Proposition 8.1, $k > l$ can be chosen so that for an arbitrary $h > k$, each ℓ_{2i+1} is homotopic rel ∂I and with values in $Q_{[l,\infty]}$ to a path in $Q_{[h,\infty]}$; and each ℓ_{2i} is homotopic rel ∂I and with values in $R_{[l,\infty]}$ to a path in $R_{[h,\infty]}$. If we have infinitely many of ℓ_i 's, only finitely many of the homotopies have to be non-identical. Hence ℓ is homotopic rel ∂I and with values in $P_{[l,\infty]}$ to a path in $P_{[h,\infty]}$. \square

Remark. The proof of Proposition 8.1 shows (see the remark to Lemma 3.11) that the images of $\pi_1(P_i)$ in $\pi_1(P_k)$ stabilize if and only if the images of $\hat{\pi}_1(P_{[i,\infty]}, X)$ in $\hat{\pi}_1(P_{[k,\infty]}, X)$ stabilize (with the same k). In this event, we call the connected compactum X *Steenrod connected over P_k* (or in more detail, *over $p_k^\infty: X \rightarrow P_k$*). It is easy to see that this property does not depend on the choice of P_{k+1}, P_{k+2}, \dots .

By Theorem 3.1(b) and Lemma 3.3, a compactum is Steenrod connected if and only if it is Steenrod connected over every map to a polyhedron. By the proof of Theorem 8.3, if X is a union of compacta Y and Z and $f: X \rightarrow P$ is a map to a polyhedron such that Y and Z are Steenrod connected over $f|_Y$ and $f|_Z$, then X is Steenrod connected over f .

An inverse sequence of groups G_i is said to be *virtually Mittag-Leffler* if for each i there exists a $j > i$ such that for each $k > j$ the image of $G_k \rightarrow G_i$ has finite index in that of $G_j \rightarrow G_i$. If X is the limit of an inverse sequence of compact connected polyhedra P_i , satisfaction of the virtual Mittag-Leffler condition for the inverse sequence $\pi_1(P_i)$ clearly does not depend on the choice of the basepoint of X and is an invariant of the equivalence relation from Proposition 2.6(iii). Thus we may call X *virtually Steenrod connected* if $\pi_1(P_i)$ are virtually Mittag-Leffler;

moreover, the property of being virtually Steenrod connected is an (unpointed) shape invariant of X .

For example, $\dots \xrightarrow{f} F_2 \xrightarrow{f} F_2$ is not virtually Mittag-Leffler, where F_2 is the free group $\langle x, y \mid \rangle$ and $f(x) = x^p$, $f(y) = y$. In particular, the wedge of the p -adic solenoid and S^1 is not virtually Steenrod connected.

Theorem 8.4. (Geoghegan–Krasinkiewicz [GK]) *If X is a connected compactum that is not virtually Steenrod connected, then $\hat{\pi}_0(X) \xrightarrow{\hat{\tau}} \pi_0(X)$ is not surjective.*

We modify the original proof in [GK], replacing certain superfluous constructions with an algebraic analysis of compactified mapping telescopes, which will be useful for the rest of this section.

Proof. Suppose that X is an inverse limit of compact connected polyhedra P_i and let $G_i = \pi_1(P_i)$. By the hypothesis, there exists a k such that the images A_i of G_{k+i} in G_k do not virtually stabilize, i.e. the pointed sets A_i/A_{i+1} are infinite for infinitely many values of i . Then if $A := \varprojlim A_0/A_i$ is endowed with the inverse limit topology (where each A_0/A_i is discrete), every compactum in A is nowhere dense in A (the details are similar to those in the proof of Lemma 3.7(b)). Since A is complete, by Baire’s Category Theorem 7.11, A is not a countable union of compacta.

The remainder of the proof is presented in two ways.

Simple argument (using Steenrod homotopy of uniform spaces, see §7). If K is a compactum, $\hat{\pi}_0(K)$ is compact since K is a continuous image of the Cantor set. Let $\tilde{X} \rightarrow X$ be the pullback of the universal covering $\tilde{P}_k \rightarrow P_k$. Then \tilde{X} is a countable union of compacta. In particular, $\hat{\pi}_0(\tilde{X})$ is a countable union of compacta.

Let \tilde{P}_{k+i} be the pullback of \tilde{P}_k over P_{k+i} . Then the inverse sequence $\dots \rightarrow \tilde{P}_1 \rightarrow \tilde{P}_0$ converges, with inverse limit \tilde{X} . It is easy to see that $\pi_0(\tilde{P}_{k+i})$ can be identified with the A_0 -set A_0/A_i , and therefore $\tilde{\pi}_0(\tilde{X}) := \varprojlim \pi_0(\tilde{P}_i)$ is A_0 -homeomorphic to A . Since $\tilde{\tau}: \pi_0(\tilde{X}) \rightarrow \tilde{\pi}_0(X)$ is a surjection similarly to Theorem 3.1(b), $\pi_0(\tilde{X})$ is not a countable union of compacta.

Hence $\hat{\tau}: \hat{\pi}_0(\tilde{X}) \rightarrow \pi_0(\tilde{X})$ is not surjective. But from the homotopy exact sequences of a Steenrod fibration (similarly to Theorem 3.15(b)) and fibration, this map is an A_0 -equivariant lift of $\hat{\tau}: \hat{\pi}_0(X) \rightarrow \pi_0(X)$. Thus the latter is not a surjection. \square

Elementary argument. The basepoint $x \in X$ gives rise to the base ray $p_{[k,\infty)}$ in $P_{[k,\infty)}$. Since each $P_{[i,\infty)}$ deformation retracts onto P_i , each A_0/A_i can be identified, as an A_0 -set, with $\pi_1(P_{[k,\infty)}, P_{[i,\infty)}; p_k)$. Similarly to the proof of Theorem 3.1(b), this yields a surjection $\varphi: G \rightarrow A$, where G is the pointed set of proper homotopy classes of proper maps $([0, \infty), 0) \rightarrow (P_{[k,\infty)}, p_k)$.²³ Then the orbit space $A_0 \backslash G$ of the left action of A_0 on G can be identified with $\pi_0(X)$.²⁴ From the homotopy exact sequence of a pair, $\hat{\pi}_0(X; x)$ is the orbit space $A_0 \backslash H$ of the left action of $A_0 \simeq \hat{\pi}_1(P_{[k,\infty)}; x)$ on $H := \hat{\pi}_1(P_{[k,\infty)}, X; x)$. Using the homotopy Π_t from §2, every $h \in H$ can be represented by a map $[-\infty, \infty] \rightarrow P_{[k,\infty]}$ sending $[-\infty, 0]$

²³By Lemma 3.13, G can be identified with $\pi_1(P_{[k,\infty)}, X)$.

²⁴In fact, it follows from the proof of Lemma 3.3 that φ is an A_0 -equivariant lift of the surjection $\varprojlim^1 G_i \rightarrow \varprojlim^1 A_i$ induced by the surjections $G_i \rightarrow A_i$.

homeomorphically onto $p_{[k,\infty]}$ and $[0,\infty)$ into $P_{[k,\infty]}$; and any two such representatives of h are homotopic through such maps. This yields an A_0 -equivariant map $\psi: H \rightarrow G$ such that the map of the orbit spaces $A_0 \backslash H \rightarrow A_0 \backslash G$ can be identified with $\hat{\tau}: \hat{\pi}_0(X) \rightarrow \pi_0(X)$. In particular, if $\hat{\tau}$ is surjective, so is ψ . Let us topologize H and G by declaring basic open sets to be the point-inverses of the maps to $\hat{\pi}_1(P_{[k,\infty]}, P_{[i,\infty]}) \simeq \pi_1(P_{[k,i]}, P_i)$, resp. to $\hat{\pi}_1(P_{[k,\infty]}, P_{[i,\infty]}) \simeq \pi_1(P_{[k,i]}, P_i)$. Then ψ and φ are continuous.

Let T_1, T_2, \dots be some triangulations of P_1, P_2, \dots such that each bonding map $P_{k+1} \rightarrow P_k$ is simplicial with respect to T_{k+1} and some subdivision of the barycentric subdivision of T_k . If Q_k denotes the dual 0-skeleton of T_k , by associating to each top-dimensional simplex σ of T_{k+1} the simplex of T_k containing the image of σ , we obtain a surjection $Q_{k+1} \rightarrow Q_k$ with finite point-inverses. The inverse limit of Q_k 's is a Cantor set C , and the inclusion $Q_{[0,\infty)} \subset P_{[0,\infty)}$ extends to a continuous map $Q_{[0,\infty)} \rightarrow P_{[0,\infty)}$ sending C onto X . Moreover, writing $K = \hat{\pi}_1(Q_{[k,\infty)} \cup P_k, C)$, it is not hard to see that the induced continuous map $\chi: K \rightarrow H$ is onto. Finally, K is the inverse limit of the discrete spaces $\pi_1(Q_{[k,i]} \cup P_k, Q_i)$, where the bonding maps have finite point-inverses. Thus K is a countable union of compacta. Since the composition $K \xrightarrow{\chi} H \xrightarrow{\psi} G \xrightarrow{\varphi} A$ is a continuous surjection, A must be a countable union of compacta, which is a contradiction. \square

Lemma 8.5. (Krasinkiewicz–Geoghegan [K3; p. 48], [GK; §7]) *Let X be the limit of an inverse sequence of compact connected polyhedra P_i and maps $p_i: P_{i+1} \rightarrow P_i$. Let $b \in X$ be the basepoint and $b_i \in P_i$ its images.*

The class α of $(g_0, g_1, \dots) \in \prod \pi_1(P_i)$ in $\varprojlim^1 \pi_1(P_i, b_i) = \pi_0(X, b)$ is represented by precisely those points of X that lie in the image of the projection of the inverse limit W_α of the universal covering spaces $\dots \xrightarrow{\tilde{p}_1} \tilde{P}_1 \xrightarrow{g_1} \tilde{P}_1 \xrightarrow{\tilde{p}_0} \tilde{P}_0 \xrightarrow{g_0} \tilde{P}_0$, where each \tilde{p}_i sends a fixed lift \tilde{b}_{i+1} of b_{i+1} into the previous one \tilde{b}_i .

In particular, α lies in the image of $\hat{\pi}_0(X, b)$ if and only if W_α is non-empty.

We call the set of points of X representing some $\alpha \in \pi_0(X)$ a *Steenrod component* of X . In this terminology, Example 5.5 and Theorem 8.4 assert existence of empty Steenrod components.

Proof. Suppose that $(g_0, g_1, \dots) \in \prod \pi_1(P_i)$ represents the image in $\pi_0(X, b)$ of some $[a] \in \hat{\pi}_0(X, b)$, where $a \in X$ is a point. Let us consider its images $a_i \in P_i$ and fix a path w_i from b_i to a_i . Then (g_0, g_1, \dots) represents the same element of $\varprojlim^1 \pi_1(P_i, b_i)$ as (g'_0, g'_1, \dots) , where g'_i is the class of the loop $w_i p_{i+1}(\bar{w}_{i+1})$. We may assume that $g'_i = g_i$ by appropriately redefining the paths w_i proceeding from the definition of \varprojlim^1 . Consider the lift \tilde{w}_i of each w_i starting at \tilde{b}_i , and let \tilde{a}_i be

its other endpoint. Then the composition $\tilde{P}_{i+1} \xrightarrow{\tilde{p}_i} \tilde{P}_i \xrightarrow{g_i} \tilde{P}_i$ sends \tilde{a}_{i+1} precisely into \tilde{a}_i . Hence the inverse limit of these compositions contains the inverse limit of the singletons $\{\tilde{a}_i\}$, which is non-empty.

Conversely, given a point \tilde{a} in the inverse limit of the compositions $\tilde{P}_{i+1} \xrightarrow{\tilde{p}_i} \tilde{P}_i \xrightarrow{g_i} \tilde{P}_i$, the preceding argument can be reversed to show that the projection a of \tilde{a} into X represents the class of (g_0, g_1, \dots) in $\pi_1(X)$. \square

Remark. Similarly to Lemma 8.5 one can prove its homological version, which can be used, for instance, to clarify the algebraic content of Examples 5.4 and 5.5. Specifically, it says that the class α of $(g_0, g_1, \dots) \in \prod H_1(P_i)$ in $\varprojlim^1 H_1(P_i) =$

$H_0(X, b)$ is represented by precisely those points of X that lie in the image of the projection of the inverse limit W_α of the universal abelian covering spaces $\dots \xrightarrow{\tilde{p}_1} \tilde{P}_1 \xrightarrow{g_1} \tilde{P}_1 \xrightarrow{\tilde{p}_0} \tilde{P}_0 \xrightarrow{g_0} \tilde{P}_0$, where each \tilde{p}_i sends a fixed lift \tilde{b}_{i+1} of b_{i+1} into the previous one \tilde{b}_i . In particular, α lies in the image of the composition $\hat{\pi}_0(X, b) \rightarrow \hat{H}_0(X) \rightarrow H_0(X)$ if and only if W_α is non-empty.

The literal converse of Theorem 8.4 does not hold:

Example 8.6 (Brin's compactum²⁵). Let $X \subset \Sigma_p \times \Sigma_p \times \Sigma_p$ be defined as the union $[\Sigma_p \times \Sigma_p \times pt] \cup [\Sigma_p \times pt \times \Sigma_p] \cup [pt \times \Sigma_p \times \Sigma_p]$. Thus $X = \lim_{\leftarrow} (\dots \xrightarrow{f} P \xrightarrow{f} P)$, where $P = [S^1 \times S^1 \times pt] \cup [S^1 \times pt \times S^1] \cup [pt \times S^1 \times S^1]$ and f is the restriction of the self-map $p \times p \times p$ of $S^1 \times S^1 \times S^1$. Since the inclusion $P \subset (S^1)^3$ induces an isomorphism on π_1 , the inclusion $X \subset (\Sigma_p)^3$ induces a bijection on π_0 . In particular, X is virtually Steenrod connected.

Let $\alpha = \overline{\dots \alpha_1 \alpha_0} \in \mathbb{Z}_p \setminus \mathbb{Z}$. We claim that $g := (\alpha + \mathbb{Z}, \alpha + \mathbb{Z}, \alpha + \mathbb{Z}) \in (\mathbb{Z}_p / \mathbb{Z})^3 \simeq \pi_0(X)$ is not in the image of $\hat{\pi}_0(X)$. Indeed, the universal cover \tilde{P} of P can be identified with the underlying space of the topological subgroup $[\mathbb{R} \times \mathbb{R} \times \mathbb{Z}] \cup [\mathbb{R} \times \mathbb{Z} \times \mathbb{R}] \cup [\mathbb{Z} \times \mathbb{R} \times \mathbb{R}]$ of \mathbb{R}^3 . The intersection of the nested sequence of cosets

$$\dots \subset (\overline{\alpha_1 \alpha_0}, \overline{\alpha_1 \alpha_0}, \overline{\alpha_1 \alpha_0}) + p^2 \tilde{P} \subset (\overline{\alpha_0}, \overline{\alpha_0}, \overline{\alpha_0}) + p \tilde{P} \subset \tilde{P}$$

is empty. Therefore by Lemma 8.5(a), g is not in the image of $\hat{\pi}_0(X)$.

Remarks. Sufficient conditions for the surjectivity of $\hat{\pi}_0(X) \xrightarrow{\hat{\tau}} \pi_0(X)$, involving the finiteness condition FP_n and n -connectedness at infinity of the fundamental groups of nerves of X , are found in [GK; §§7,9]. According to [GK; Remark 10.3], Ferry proved the following converse of Theorem 8.4: every virtually Steenrod connected compactum (in particular, Brin's compactum!) is Steenrod homotopy equivalent to a compactum X such that $\hat{\pi}_0(X) \rightarrow \pi_0(X)$ is surjective.

Theorem 8.7. (Krasinkiewicz–Minc [KM]) *Let X be a connected, Steenrod disconnected compactum. Then the image of $\hat{\tau}: \hat{\pi}_0(X) \rightarrow \pi_0(X)$ is uncountable (in particular, nontrivial).*

We substantially simplify the original proof [KM; pp. 147–151] (continued from [K2; pp. 146–147]), which depended on the theory of continua (essentially using composants and indecomposable continua). Note that Theorem 8.7 implies Theorems 8.2 and 8.3.

Proof. Let $\dots \xrightarrow{p_2} P_2 \xrightarrow{p_1} P_1$ be an inverse sequence of polyhedra with $\lim_{\leftarrow} P_i = X$ and such that X is Steenrod disconnected over P_1 (see the definition in the remark to Theorem 8.3). Let G_i be the image of $\pi_1(P_i)$ in $\pi_1(P_1)$. Let BG_1 be a classifying space for $\pi_1(P_1)$ (a locally compact polyhedron of the homotopy type of $K(G_1, 1)$), and let BG_i be the covering space of BG_1 corresponding to G_i . Then the bonding maps $p_1^i: P_i \rightarrow P_1$ lift to maps $f_i: P_i \rightarrow BG_i$ commuting with the p_i and with the covering maps $q_i: BG_{i+1} \rightarrow BG_i$. These converge to a map f from X into the inverse limit BG of BG_i 's. Let Q_i be the union of all (closed) simplices of BG_i that intersect $f_i(P_i)$, where BG_i is triangulated arbitrarily for $i = 1$ and so that q_{i-1} is simplicial for $i > 1$. Note that each Q_i is compact since $f_i(P_i)$ is. Each inclusion

²⁵This may or may not be the actual Brin's compactum mentioned in [GK; Remark 10.4].

induced map $\pi_1(Q_i) \rightarrow G_i$ is an epimorphism since f_i factors through Q_i . Hence $Y := \varprojlim Q_i$ is Steenrod disconnected over Q_1 .

By Lemma 8.5, each Steenrod component α of BG is the image of an inverse limit W_α of self-homeomorphisms of the universal cover EG_1 of BG_1 . If EG_1 is triangulated so that the projection $EG_1 \rightarrow BG_1$ is simplicial, we obtain that α is a continuous image of the triangulated polyhedron W_α , moreover the composition $W_\alpha \rightarrow \alpha \subset BG \rightarrow BG_i$ is simplicial. In particular, it follows that α contains only countably many points of the inverse limit $BG^{(0)}$ of the 0-skeleta $BG_i^{(0)}$. On the other hand, each point of the inverse limit $Y^{(0)}$ of the 0-skeleta $Q_i^{(0)}$ is contained in the same Steenrod component of BG as some point of $f(X)$. Indeed, the “closed star” S of each vertex $y \in Y^{(0)}$ (i.e. the image of the star of a lift of y into the triangulated polyhedron W_α , where $y \in \alpha$) is the inverse limit of the stars S_i of $q_i^\infty(y)$. Each $S_i \cap f_i(P_i)$ is nonempty, hence so is their inverse limit $S \cap f(X)$.

Suppose that $Y^{(0)}$ is countable. Then by the Baire Category Theorem 7.11 it contains an isolated point v . Removing the “open star” of the vertex v from $Y_0 := Y$, we obtain a compactum that has finitely many connected components by the cohomological Mayer–Vietoris sequence (using that the “link” of v is a compact polyhedron). At least one of them is Steenrod disconnected over Q_0 by the remark to Theorem 8.3. This component Y_1 has the 0-skeleton $Y_1^{(0)} := Y_1 \cap Y^{(0)}$, which again contains an isolated point, so that the argument can be repeated. Then $Y_\omega := \bigcap_{i < \omega} Y_i$ is the inverse limit of the inclusions $\dots \subset Y_1 \subset Y_0$. Hence Y_ω is connected (by the continuity of Pontryagin cohomology) and homeomorphic to the limit of the ‘staircase’ inverse sequence $\dots \xrightarrow{p_{11}} P_{11} \xrightarrow{f_{11}} P_{01} \xrightarrow{p_{00}} P_{00}$, where each $Y_i = \varprojlim (\dots \xrightarrow{p_{i1}} P_{i1} \xrightarrow{p_{i0}} P_{i0} = P)$, and each $f_{i,[0,\infty)}: P_{i+1,[0,\infty)} \rightarrow P_{i,[0,\infty)}$ is the level-preserving map representing the Steenrod homotopy class of the inclusion $Y_{i+1} \subset Y_i$. Hence each $\Gamma_i := \text{im}[\pi_1(P_{ii}) \rightarrow \pi_1(P)]$ properly contains Γ_{i+1} , so Y_ω is Steenrod disconnected over Q_0 .²⁶

Finally, $Y_\omega^{(0)} := Y_\omega \cap Y^{(0)}$ is again countable, and the process can be continued by transfinite (countable) induction until it stops, i.e. until we reach the stage λ where Y_λ is empty. This is a contradiction, since the empty set Y_λ cannot be Steenrod disconnected over Q_0 .

Thus $Y^{(0)}$ is uncountable. Since each Steenrod component of BG contains only countably many points of $BG^{(0)}$, we obtain that $Y^{(0)}$ meets uncountably many Steenrod components of BG . Since each point of $Y^{(0)}$ is contained in the same Steenrod component of BG as some point of $f(X)$, the image of the composition $\hat{\pi}_1(X) \rightarrow \hat{\pi}_1(BG) \rightarrow \pi_1(BG)$ is uncountable. Since this composition factors through $\pi_1(X)$, the image of $\hat{\tau}: \hat{\pi}_1(X) \rightarrow \pi_1(X)$ is uncountable. \square

Remark. The proof works if the map $f: X \rightarrow BG$ is replaced by a map of $Z := \varprojlim \Gamma_i/G_1$ into BG , where each Γ_i is the Cayley graph of the action of a certain finite set of generators of G_i on G_1 (which is disconnected unless $G_i = G_1$). In more detail, Z can be constructed as follows. Let $D \subset EG_1$ be a (closed) fundamental domain of the action of G_1 . Let $\tilde{X} \rightarrow X$ be the pullback of $EG_1 \rightarrow BG_1$. Since $\tilde{X} \rightarrow EG_1$ factors through the universal cover \tilde{P}_1 of P_1 , and $\tilde{X} \rightarrow \tilde{P}_1$ is proper, the

²⁶Note that the intersection of a nested sequence of Steenrod disconnected compacta does not have to be Steenrod disconnected. For instance, the intersection of a nested sequence of clusters of solenoids can be a single point.

fundamental domain \hat{D} of the action of G_1 on \tilde{X} is compact. Then $\tilde{\pi}_0(\hat{D})$ is also compact, and so is its image $Z^{(0)}$ in $\tilde{\pi}_0(\tilde{X}) := \varprojlim \pi_0(\tilde{P}_i)$. As G_1 acts on $\tilde{\pi}_0(\tilde{X})$, each generator g_i of G_1 sends $Z^{(0)}$ to its copy $g_i Z^{(0)}$. In particular, this yields a map $Z^{(0)} \cap g_i^{-1} Z^{(0)} \rightarrow Z^{(0)}$, which may be viewed as a partial self-map h_i of $Z^{(0)}$. Let Z be the union of the mapping tori of all h_i corresponding to a fixed collection of generators of G_1 ; since this collection may be assumed finite, Z is compact.

Remark. Theorem 8.7 has a homological analogue: If X is connected but $H_0(X) \neq 0$, then the image of $\hat{\tau}: \hat{H}_0(X) \rightarrow H_0(X)$ is uncountable. Compared with Theorem 8.7, this assertion is much easier; it can be verified by comparing X with an l -adic solenoid (l is a sequence of primes). We leave the details to the reader.

We are now ready for the promised 0-dimensional counterpart of Corollary 6.6:

Corollary 8.8. *A compactum X is semi- LC_0 iff $\pi_0(X)$ is discrete.*

Proof. The “if” part follows from Lemma 3.4(b). Conversely, let us fix some metric on X , and let $\varepsilon > 0$ be such that every two ε -close points of X represent the same Steenrod homotopy class. Obviously, $\tilde{\pi}_0(X)$ is discrete. If C is a component of X , any pair of points $p, q \in C$ is connected by a chain $p = p_0, p_1, \dots, p_n = q$, where each p_{i+1} is ε -close to p_i . Hence all points of C represent the same Steenrod homotopy class. By Theorem 8.7, C is Steenrod connected. \square

Remark. By regarding $\hat{\pi}_n(X)$ as $\hat{\pi}_0(\Omega^n X)$, where $\Omega^n X$ is the iterated loop space of X (which is a separable metrizable complete uniform space), the proof of Theorem 8.7 can apparently be used to prove that if X is an LC_{n-1} and UV_1 compactum and $\ker[\pi_n(X) \xrightarrow{\hat{\tau}} \tilde{\pi}_n(X)]$ is nontrivial, then the image of $\ker[\hat{\pi}_n(X) \xrightarrow{\hat{\tau}\hat{\tau}} \tilde{\pi}_n(X)]$ in $\pi_n(X)$ is uncountable. (When $n \geq 2$, this assertion also follows from Theorems 6.5 and 6.1 and the uncountability of every nontrivial derived limit of countable groups, which in turn follows from Lemma 3.3.) The details will hopefully appear elsewhere (see also Lemma 7.9(g)).

Theorem 8.9. *A connected compactum X is Steenrod connected iff every overlay space over X has countably many uniform components.*

The proof continues the analysis begun in the proof of Theorem 8.7.

Proof. Let us fix a map $f: X \rightarrow P$, where P is a polyhedron. We will show that X is Steenrod connected over P (see the definition in the remark after Theorem 8.3) iff the pullback \tilde{X} of the universal cover \tilde{P} of P has countably many uniform components.

We may assume that $X = \varprojlim (\dots \rightarrow P_2 \rightarrow P_1 = P)$, where the P_i are compact polyhedra and the bonding maps are PL. Let \tilde{P}_i be the pullback of \tilde{P} over P_i . Then $G := \pi_1(P)$ acts on the components of \tilde{P}_i with stabilizer $\text{im}(G_i \rightarrow G)$, where $G_k = \pi_1(P_i)$. If X is Steenrod connected over P , let k be such that $\text{im}(G_k \rightarrow G) = \text{im}(G_i \rightarrow G)$ for each $i > k$. Then the G -equivariant map $\pi_0(\tilde{P}_i) \rightarrow \pi_0(\tilde{P}_k)$ is a bijection for each $i > k$. Hence $\tilde{\pi}_0(\tilde{X})$ is countable. On the other hand, since $\dots \rightarrow P_2 \rightarrow P_1$ is convergent (see §7), so is $\dots \rightarrow \tilde{P}_2 \rightarrow \tilde{P}_1$. Now \tilde{X} is the uniformly disjoint union of the preimages \tilde{X}^c of the components \tilde{P}_k^c of \tilde{P}_k . Hence if \tilde{P}_i^c denotes the corresponding component of \tilde{P}_i for $i > k$, the inverse sequence $\dots \rightarrow \tilde{P}_{k+1}^c \rightarrow \tilde{P}_k^c$ is convergent. Then by Lemma 7.9(f), its inverse limit \tilde{X}^c is uniformly connected. Thus the set of uniform components of \tilde{X} injects into $\tilde{\pi}_0(\tilde{X})$.

Conversely, suppose that X is Steenrod disconnected over P ; we shall use the notation in the proof of Theorem 8.7. The universal covering $\tilde{P} \rightarrow P$ is induced from $EG_1 \rightarrow BG_1$ by f_1 . Let \tilde{Q}_i and \tilde{Y} be the pullbacks of EG_1 over Q_i and Y . Since each Q_i is embedded into the covering space $(q_1^i)^{-1}(Q_1) \subset BG_i$ of Q_1 , the map $\tilde{Q}_i \rightarrow \tilde{Q}_1$ embeds each component of \tilde{Q}_i . On the other hand, since $Y^{(0)}$ has been proved to be uncountable, there exists an $x \in Q_1^{(0)}$ whose preimage $(q_1^\infty)^{-1}(x)$ in $Y^{(0)}$ is uncountable. Pick a lift $\tilde{x} \in \tilde{Q}_1$; then its preimage in \tilde{Y} is uncountable, and at the same time the points of its preimage in each \tilde{Q}_i lie in distinct components of \tilde{Q}_i . Thus \tilde{Y} has uncountably many uniform components. However, by the proof of Theorem 8.7, $f: X \rightarrow Y$ induces a surjection on path components. Therefore so does its lift $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$. \square

Remark. Theorem 8.9 implies Theorem 8.7, because if $f: E \rightarrow B$ is an overlaying, every point-inverse of $f_*: \pi_0(E) \rightarrow \pi_0(B)$ is countable. Indeed, the point-inverse of the Steenrod homotopy class of the basepoint is countable similarly to Theorem 3.15(b); and by the proof of 3.15(b) we are free to choose any Steenrod homotopy class $b: pt \rightsquigarrow B$ that factors into composition of some $\tilde{b}: pt \rightarrow E$ and the overlaying f as our “basepoint”.

As remarked in §3, an inverse limit of fibrations $E_i \rightarrow B_i$ need not be a fibration; however, an inverse limit of compositions f_i of fibrations with the pullbacks of the preceding compositions f_{i-1} is a fibration. In particular, an inverse limit of coverings is a fibration. (This does not depend on whether the inverse sequence of the total spaces is convergent.) On the other hand, by definition (see §3), the limit of an inverse sequence of fibrations over compact polyhedra is a Steenrod fibration provided that the inverse sequence of the total spaces is convergent (see §7). The following example shows that the latter assumption cannot be dropped.

Example 8.10. Let $X \subset \Sigma_p \times \Sigma_p \times I$ be defined as the union $[\Sigma_p \times \Sigma_p \times \partial I] \cup [\Sigma_p \times pt \times I] \cup [pt \times \Sigma_p \times I]$. Thus $X = \varprojlim (\dots \xrightarrow{p_1} P_1 \xrightarrow{f} P_0)$, where each $P_i = [S^1 \times S^1 \times \partial I] \cup [S^1 \times pt \times I] \cup [pt \times S^1 \times I]$ and each f_i is the restriction of the self-map $p \times p \times \text{id}_I$ of $S^1 \times S^1 \times I$. Let us consider the Steenrod component $h = (\alpha + \mathbb{Z}, \alpha + \mathbb{Z})$ in $(\mathbb{Z}_p/\mathbb{Z})^2 \simeq \pi_0(X)$, where $\alpha = \overline{\dots \alpha_1 \alpha_0} \in \mathbb{Z}_p \setminus \mathbb{Z}$ (compare Example 8.6). Let $X^h \subset X$ be the set of its points, i.e. the union of all path components of X that map into h . Let $\varphi: \tilde{X} \rightarrow X$ be the overlaying, induced from the universal covering $\tilde{P}_1 \rightarrow P_1$, and let $\varphi^h: \tilde{X}^h \rightarrow X^h$ denote its restriction to $\tilde{X}^h := \varphi^{-1}(X^h)$.

We claim that φ^h is not a Steenrod fibration. Indeed, let us consider the pullbacks $\tilde{P}_i \rightarrow P_i$ of the universal covering of P_1 . We can identify \tilde{X}^h with the inverse limit $\mathbb{R} \times \mathbb{R} \times \partial I$ of the subpolyhedra $\tilde{P}_i^h := [\mathbb{R} \times \mathbb{R} \times \partial I] \cup [K_i \times I]$ of the \tilde{P}_i , where $K_i = [(\overline{\alpha_{i-1} \dots \alpha_0} + p^i \mathbb{Z}) \times \mathbb{R}] \cup [\mathbb{R} \times (\overline{\alpha_{i-1} \dots \alpha_0} + p^i \mathbb{Z})]$. Note that $\tilde{b} := (0, 0, 0)$ and $\tilde{b}' := (0, 0, 1)$ belong to different components of \tilde{X}^h but the corresponding proper rays $\tilde{b}_{[0, \infty)}$ and $\tilde{b}'_{[0, \infty)}$ are properly homotopic in the mapping telescope $\tilde{P}_{[0, \infty)}^h$. This proper homotopy projects to a Steenrod path $\ell: (I, \partial I) \times [0, \infty) \rightarrow (P_{[0, \infty)}, b_{[0, \infty)} \cup b'_{[0, \infty)})$, where $b = \varphi^h(\tilde{b})$ and $b' = \varphi^h(\tilde{b}')$.

If φ^h is a Steenrod fibration, there exists a convergent (see §7) inverse sequence $\dots \rightarrow \hat{P}_2^h \rightarrow \hat{P}_1^h$ with $\varprojlim \hat{P}_i^h = \tilde{X}^h$ and a level-preserving map $w_{[0, \infty)}: \hat{P}_{[0, \infty)}^h \rightarrow P_{[0, \infty)}$ with $\varprojlim w_i = \varphi^h$ such that, by the proof of Theorem 3.15(b), $p_{[-3, \infty-3)}^{[0, \infty)} \ell$

lifts to a Steenrod path $\tilde{\ell}: I \times [0, \infty) \rightarrow \hat{P}_{[0, \infty)}^h$ starting at $\hat{b}_{[0, \infty)}$, where \hat{b}_i denotes the image of \tilde{b} in \hat{P}_i^h . This $\tilde{\ell}$ has to end at $\hat{b}_{[0, \infty)}''$, where \hat{b}_i'' denotes the image in \hat{P}_i^h of some lift $\tilde{b}'' \in \tilde{X}^h$ of b' . Then \tilde{b}'' and \tilde{b} belong to the same Steenrod component of $\tilde{X}^h = \mathbb{R} \times \mathbb{R} \times \partial I$, which is a contradiction since $\tilde{b} = (0, 0, 0)$ and $\tilde{b}' = (m, n, 1)$ for some $m, n \in \mathbb{Z}$.

Remark. The above example also shows that the inclusion induced maps $\pi_0(\tilde{X}^h) \rightarrow \pi_0(\tilde{X})$ and $\tilde{\pi}_0(\tilde{X}^h) \rightarrow \tilde{\pi}_0(\tilde{X})$ need not be injective. This may be of interest in connection with the proof of the “easy” (only if) direction in Theorem 8.9.

REFERENCES

- [AgS] V. V. Agaronjan, Yu. M. Smirnov, *The shape theory for uniform spaces and the shape uniform invariants*, Comment. Math. Univ. Carolin. **19** (1978), 351–357.
- [AM] M. Artin, B. Mazur, *Etale Homotopy*, Lecture Notes in Math., vol. 100, Springer, 1969.
- [AS] M. Atiyah, G. B. Segal, *Equivariant K-theory and its completion*, J. Diff. Geom. **3** (1969), 1–18.
- [BB] I. K. Babenko, S. A. Bogatyĭ, *On the group of substitutions of formal power series with integer coefficients*, Izvestia RAN **72** (2008), no. 2, 39–64; English transl., Izv. Math. **72** (2008), 241–264.
- [BM] M. G. Barratt, J. Milnor, *An example of anomalous singular homology*, Proc. Amer. Math. Soc. **13** (1962), 293–297.
- [Bog] S. Bogatyĭ, *The Vietoris theorem for shapes, inverse limits, and a certain problem of Ju. M. Smirnov*, Dokl. Akad. Nauk SSSR **211** (1973), 764–767; English transl., Soviet Math. Dokl. **14** (1973), 1089–1094.
- [BoM] A. Borel, J. C. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. **7** (1960), 137–159.
- [Bo] K. Borsuk, *On the n -movability*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **20** (1972), 859–864.
- [BK] A. K. Bousfield, D. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math., vol. 304, Springer, 1972.
- [Br] G. E. Bredon, *Sheaf Theory (2nd ed.)*, Grad. Texts in Math., vol. 170, Springer, 1997.
- [BH] M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer, Berlin, 1999.
- [BDLM] N. Brodskiy, J. Dydak, B. Labuz, A. Mitra, *Rips complexes and covers in the uniform category*, arXiv:0706.3937.
- [Bu] V. M. Bukhshtaber, *Groups of polynomial transformations of a line, non-formal symplectic manifolds, and the Landweber–Novikov algebra*, Russ. Math. Surv. **54** (1999), 837–838.
- [BSh] V. M. Bukhshtaber, A. V. Shokurov, *The Landweber–Novikov algebra and formal vector fields on the line*, Funkt. Anal. Appl. **12** (1978), 159–168.
- [BCSS] R. Budney, J. Conant, K. P. Scannell, D. Sinha, *New perspectives on self-linking*, Adv. Math. **191** (2005), 78–113; arXiv:math/0303034.
- [BRS] S. Buoncrisiano, C. P. Rourke, B. J. Sanderson, *A Geometric Approach to Homology Theory*, London Math. Soc. Lecture Note Ser., vol. 18, Cambridge Univ. Press, 1976.
- [C1] F. Cathey, *Strong shape theory*, Shape Theory and Geometric Topology (Proceedings, Dubrovnik 1981), Lecture Notes in Math., vol. 870, Springer, 1981, pp. 215–238.
- [C2] ———, *Shape fibrations and strong shape theory*, Topol. Appl. **14** (1982), 13–30.
- [Č] E. Čech, *Théorie générale de l’homologie dans un espace quelconque*, Fund. Math. **19** (1932), 149–183.
- [CS] T. A. Chapman, L. C. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, Acta Math. **137** (1976), 171–208.
- [Ch] D. E. Christie, *Net homotopy for compacta*, Trans. Amer. Math. Soc. **56** (1944), 275–308.
- [Do] D. Doitchinov, *Uniform shape and uniform Čech homology and cohomology groups for metric spaces*, Fund. Math. **102**, 209–218.
- [D1] J. Dydak, *Concerning the abelianization of the first shape group of pointed continua*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **20** (1972), 859–865.

- [D2] ———, *The Whitehead and Smale theorems in shape theory*, Dissertationes Math. (Rozprawy Mat.) **87** (1976), PWN (Polish Sci. Publ.), Warszawa.
- [D3] ———, *A simple proof that pointed FANR-spaces are regular fundamental retracts of ANR's*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **25** (1977), 55–62.
- [D4] ———, *Relations between homology and homotopy pro-groups of continua*, Topol. Proc. **6** (1981), 267–278.
- [D5] ———, *Local n -connectivity of quotient spaces and one-point compactifications*, Shape Theory and Geometric Topology (Proc., Dubrovnik, 1981), Lecture Notes in Math., vol. 870 Springer, 1981, pp. 48–72.
- [D6] ———, *Steenrod homology and local connectedness*, Proc. Amer. Math. Soc. **98** (1986), 153–157.
- [DS1] J. Dydak, J. Segal, *Shape Theory*, Lecture Notes in Math., vol. 688, Springer-Verlag, 1978.
- [DS2] ———, *Strong shape theory*, Dissertationes Math. (Rozprawy Mat.) **192** (1981), PWN (Polish Sci. Publ.), Warszawa; announced in, *Strong shape theory: a geometrical approach*, Topol. Proc. **3** (1978), 59–72; surveyed in, J. Dydak, *Strong shape theory, a survey of results*, Proc. Int. Conf. on Geometric Topology (Warszawa 1978), PWN (Polish Sci. Publ.), Warszawa, 1980.
- [DS3] ———, *Local n -connectivity of decomposition spaces*, Topol. Appl. **18** (1984), 43–58.
- [EG1] D. A. Edwards, R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction*, Ann. Math. **101** (1975), 521–535.
- [EG2] ———, *The stability problem in shape, and a Whitehead theorem in pro-homotopy*, Trans. Amer. Math. Soc. **214** (1975), 261–277.
- [EH] D. A. Edwards, H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math., vol. 542, Springer, 1976.
- [EKR] K. Eda, U. Karimov and D. Repovš, *On (co)homology locally connected spaces*, Topol. Appl. **120** (2002), 397–401.
- [EK1] K. Eda, K. Kawamura, *The surjectivity of the canonical homomorphism from singular homology to Čech homology*, Proc. Amer. Math. Soc. **128** (1999), 1487–1495.
- [EK2] ———, *The singular homology of the Hawaiian earring*, J. London Math. Soc. **62** (2000), 305–310.
- [EK3] ———, *Homotopy and homology groups of the n -dimensional Hawaiian earring*, Fund. Math. **165** (2000), 17–28.
- [ES] S. Eilenberg, N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, 1952.
- [Fe1] S. Ferry, *Homotoping ε -maps to homeomorphisms*, Amer. J. Math. **101** (1979), 567–582.
- [Fe2] ———, *A stable converse to the Vietoris–Smale theorem with applications to shape theory*, Trans. Amer. Math. Soc. **261** (1980), 369–386.
- [Fe3] ———, *Remarks on Steenrod homology, Novikov Conjecture, Index Theorems and Rigidity* (vol. 2), London Math. Soc. Lecture Note Ser., vol. 227, Cambridge Univ. Press, 1995, pp. 149–166; <http://www.maths.ed.ac.uk/~aar/>.
- [FR] S. Ferry, A. Ranicki, *A survey of Wall's finiteness obstruction*, Surveys on Surgery Theory (vol. 2), Princeton Univ. Press, 2000, pp. 63–80; <http://www.maths.ed.ac.uk/~aar/>.
- [F1] R. H. Fox, *On shape*, Fund. Math. **74** (1972), 47–71.
- [F2] ———, *Shape theory and covering spaces*, Topology Conference (Virginia Polytech. Inst. and State Univ., 1973), Lecture Notes in Math., vol. 375, Springer, 1974, pp. 71–90.
- [GHW] D. J. Garity, J. P. Henderson, D. G. Wright, *Menger spaces and inverse limits*, Pacific J. Math. **131** (1988), 249–259.
- [Ge1] R. Geoghegan, *A note on the vanishing of \lim^1* , J. Pure Appl. Algebra **17** (1980), 113–116.
- [Ge2] ———, *Topological Methods in Group Theory*, Grad. Texts in Math., vol. 243, Springer, 2008.
- [GK] R. Geoghegan, J. Krasinkiewicz, *Empty components in strong shape theory*, Topol. Appl. **41** (1991), 213–233.
- [G] B. I. Gray, *Spaces of the same n -type, for all n* , Topology **5** (1966), 241–243.
- [Gr] J. Grossman, *Homotopy classes of maps between pro-spaces*, Michigan Math. J. **21** (1974), 355–362.

- [Gü] B. Günther, *Semigroup structures on derived limits*, J. Pure Appl. Algebra **69** (1990), 51–65.
- [Gu] M. N. Gusarov, *Variations of knotted graphs. Geometric techniques of n -equivalence*, Algebra i Analiz **12** (2000), no. 4, 79–125; English transl., St.-Petersburg Math. J. **12** (2000), 569–604; <http://www.math.toronto.edu/~drorbn/Goussarov/>.
- [Ha] K. Habiro, *Claspers and finite type invariants of links*, Geom. Topol. **4** (2000), 1–83; [arXiv:math/0001185](https://arxiv.org/abs/math/0001185).
- [He] L. J. Hernández-Paricio, *Fundamental pro-groupoids and covering projections*, Fund. Math. **156** (1998), 1–31.
- [Hu] W. Hurewicz, *Homologie, Homotopie und lokaler Zusammenhang*, Fund. Math. **25** (1935), 467–485.
- [Ir] K. Iriye, *The first derived functor of the inverse limit and localization*, J. Pure Appl. Algebra **173** (2002), 7–14.
- [Is] J. R. Isbell, *Uniform Spaces*, Amer. Math. Soc., 1964.
- [IS] Y. Iwamoto and K. Sakai, *Strong n -shape theory*, Topol. Appl. **122** (2002), 253–267.
- [Jo] D. L. Johnson, *The group of formal power series under substitution*, J. Aust. Math. Soc., Ser. A **45** (1988), 296–302.
- [Ju] O. Jussila, *On homology theories in locally connected spaces*, Ann. Acad. Sci. Fenn. Ser. A (1964), no. 340, 15pp; *II*, *ibid.* (1965), no. 378, 10pp.
- [Ke] J. Keesling, *Algebraic invariants in shape theory*, Topol. Proc. **1** (1976), 115–124.
- [Ko1] Y. Kodama, *On embeddings of spaces into ANR and shapes*, J. Math. Soc. Japan **27** (1975), 533–544.
- [Ko2] ———, *Fine movability*, J. Math. Soc. Japan **30** (1978), 101–116.
- [Ko3] ———, *A characteristic property of a finite-dimensional pointed FANR*, Japan J. Math. **4** (1978), 445–460.
- [Ko4] ———, *Generalization of movability and Hurewicz’s isomorphism theorem for Steenrod homology*, Russian Math. Surveys **34** (1979), no. 6, 57–59.
- [KK] Y. Kodama, A. Koyama, *Hurewicz isomorphism theorem for Steenrod homology*, Proc. Amer. Math. Soc. **74** (1979), 363–367.
- [KO] Y. Kodama, J. Ono, *On fine shape theory*, Fund. Math. **105** (1979/80), 29–39; *II*, Fund. Math. **108** (1980), 89–98; Y. Kodama, *III*, Glas. Mat. **16** (1981), 369–375.
- [KOW] Y. Kodama, J. Ono, T. Watanabe, *AR associated with ANR-sequence and shape*, Gen. Topol. Appl. **9** (1978), 71–88.
- [Koy] A. Koyama, *A Whitehead-type theorem in fine shape theory*, Glas. Mat. **18** (1983), 359–370.
- [Koyt] R. Koytcheff, *A homotopy-theoretic view of Bott–Taubes integrals and knot spaces*, [arXiv:0810.1785](https://arxiv.org/abs/0810.1785).
- [K1] J. Krasinkiewicz, *On a method of constructing ANR-sets. An application of inverse limits*, Fund. Math. **92** (1976), 95–112.
- [K2] ———, *Continuous images of continua and 1-movability*, Fund. Math. **98** (1978), 141–164.
- [K3] ———, *On pointed 1-movability and related notions*, Fund. Math. **114** (1981), 29–52.
- [KM] J. Krasinkiewicz, P. Minc, *Generalized paths and pointed 1-movability*, Fund. Math. **104** (1979), 141–153.
- [Ku1] K. Kuperberg, *Two Vietoris-type isomorphism theorems in Borsuk’s theory of shape, concerning the Vietoris–Čech homology and Borsuk’s fundamental groups*, Studies in Topology, 1975, pp. 285–314.
- [Ku2] ———, *A note on the Hurewicz isomorphism theorem in Borsuk’s theory of shape*, Fund. Math. **90** (1976), 173–175.
- [LaB] B. LaBuz, *Inverse limits of uniform covering maps*, [arXiv:0808.4119](https://arxiv.org/abs/0808.4119).
- [La] R. C. Lacher, *Cell-like spaces*, Proc. Amer. Math. Soc. **20** (1969), 598–602.
- [Le] S. Lefschetz, *Algebraic Topology*, Amer. Math. Soc., 1942.
- [Li] Yu. T. Lisitsa, *Hurewicz and Whitehead theorems in the strong shape theory*, Dokl. Akad. Nauk SSSR **283** (1985), no. 1, 38–43; English transl., Soviet Math. Dokl. **32** (1985), no. 1, 36–39.
- [Ma] S. Mardešić, *Comparison of singular and Čech homology in locally connected spaces*, Michigan Math. J. **6** (1959), 151–166.

- [MaM] S. Mardešić, V. Matijević, *Classifying overlay structures of topological spaces*, Topol. Appl. **113** (2001), 167–209.
- [MaS] S. Mardešić, J. Segal, *Shape Theory (The Inverse System Approach)*, North Holland, Amsterdam, 1982.
- [MR] S. Mardešić, T. B. Rushing, *Shape fibrations I*, Gen. Topol. Appl. **9** (1978), 193–215.
- [Mas] W. S. Massey, *Homology and Cohomology Theory*, Marcel Dekker, New York, 1978; the author's guide for reading this book, *How to give an exposition of the Čech–Alexander–Spanier type homology theory*, Amer. Math. Monthly **85** (1978), 75–83.
- [MM] C. A. McGibbon and J. M. Møller, *On spaces with the same n -type for all n* , Topology **31** (1992), 177–201.
- [Mc] D. R. McMillan, *One-dimensional shape properties and three-manifolds*, Studies in Topology (Proc., Charlotte, 1974), Academic Press, New York, 1975, pp. 367–381.
- [M1] S. Melikhov, *A multivariable analogue of the Conway polynomial*, arXiv:math/0312007.
- [M2] ———, *Isotopic and continuous realizability of maps in the metastable range*, Mat. Sbornik **195:7** (2004), 71–104; English transl., Sb. Math. **195** (2004), 983–1016.
- [M3] ———, *A polynomial compactification of configuration spaces and resolution of the Thom–Boardman singularities* (in preparation).
- [MS] S. A. Melikhov, E. V. Shchepin, *The telescope approach to embeddability of compacta*, arXiv:math/0612085.
- [Mi] J. Milnor, *On the Steenrod homology theory* (1961), Berkeley; Novikov Conjectures, Index Theorems and Rigidity (vol. 2), London Math. Soc. Lecture Note Ser., vol. 226, Cambridge Univ. Press, 1995, pp. 79–96; <http://www.maths.ed.ac.uk/~aar/>.
- [MRS] W. J. R. Mitchell, D. Repovš, E. V. Ščepin, *On 1-cycles and the finite dimensionality of homology 4-manifolds*, Topology **31** (1992), 605–623.
- [Miy] T. Miyata, *Homology, cohomology, and uniform shape*, Glasnik Mat. **30** (1995), 85–109.
- [Mo] T. T. Moore, *On Fox's theory of overlays*, Fund. Math. **99** (1978), 205–211.
- [P] L. S. Pontrjagin, *Über den algebraischen Inhalt topologischer Dualitätssätze*, Math. Ann. **105** (1931), 165–205; Russian transl., Izbrannye nauchnye trudy, vol. 1, Nauka, Moscow, 1988, pp. 65–105.
- [Po] T. Porter, *Čech and Steenrod homotopy and the Quigley exact couple in strong shape and proper homotopy theory*, J. Pure Appl. Algebra **24** (1982), 303–312.
- [Q1] J. B. Quigley, *An exact sequence from the n th to the $(n-1)$ st fundamental group*, Fund. Math. **77** (1973), 195–210.
- [Q2] ———, *Equivalence of fundamental and approaching groups of movable pointed compacta*, Fund. Math. **91** (1976), 73–83.
- [Ru] V. Runde, *A Taste of Topology*, Springer, Berlin, 2005.
- [SSG] J. Segal, S. Spiez, B. Günther, *Strong shape of uniform spaces*, Topol. Appl. **49** (1993), 237–249.
- [Sh] N. Shrikhande, *Homotopy properties of decomposition spaces*, Fund. Math. **66** (1983), 119–124.
- [S1] L. C. Siebenmann, *Infinite simple homotopy types*, Indag. Math. **32** (1970), 479–495.
- [S2] ———, *Chapman's classification of shapes: a proof using collapsing*, Manuscripta Math. **16** (1975), 373–384.
- [Si] D. Sinha, *The topology of spaces of knots*, arXiv:math/0202287.
- [Sk] E. G. Sklyarenko, *Homology and cohomology of general spaces*, Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya. Obshchaja topologija — 2, vol. 50, 1989, pp. 129–266; <http://www.mathnet.ru>; English transl., General Topology II. Encycl. Math. Sci., vol. 50, 1996, pp. 119–246.
- [Sp] E. Spanier, *Algebraic topology*, McGraw–Hill, New York, 1966.
- [St] N. E. Steenrod, *Regular cycles of compact metric spaces*, Ann. Math. **41** (1940), 833–851.
- [Vi] J. Vilímovský, *Uniform quotients of metrizable spaces*, Fund. Math. **127** (1987), 51–55.
- [Vo] R. M. Vogt, *On the dual of a lemma of Milnor*, Proc. of the Advanced Study Institute on Algebraic Topology (1970), Vol. III, Various Publ. Ser., No. 13, Mat. Inst., Aarhus Univ., Aarhus, 1970, pp. 632–648.
- [V1] I. Volić, *Configuration space integrals and Taylor towers for spaces of knots*, Topol. Appl. **153** (2006), 2893–2904; arXiv:math/0401282.
- [V2] ———, *Finite type knot invariants and calculus of functors*, Compos. Math. **142** (2006), 222–250; arXiv:math/0401440.

- [Wa] T. Watanabe, *On a problem of Y. Kodama*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), 981-985.
- [Wh] J. H. C. Whitehead, *A certain exact sequence*, Ann. Math. **52** (1950), 51-110.
- [Wi] R. E. Williamson, *Cobordism of combinatorial manifolds*, Ann. Math. **83** (1966), 1-33.
- [Zd] S. Zdravkovska, *An example in shape theory*, Proc. Amer. Math. Soc. **83** (1981), 594-596.